High-Dimensional Expanders imply Agreement Expanders

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1 First Session

We present the following result of Dinur and Kaufman.

Theorem 1.1 (Informal Main [1]). *High-dimensional expanders imply* ***derandomized*** *agreement expanders.*

We will see that the theorem above is a derandomized direct product test which is a known test in the PCP literature [3, 2]. In this first session, we will elaborate on the objects and terms appearing in this theorem. We start with the notion of *agreement expansion*, but first we need to introduce agreement tests (tailored to our applications).

1.1 Agreement Tests and Agreement Expanders

Let $V = [n]$ and $X \subseteq 2^V$. We say that $F = \{f_S\}_{S \in X}$, where each $f_S \in \{0,1\}^S$, is an *ensemble* on the collection X. A natural question is the following.

Question 1.2. *Given an ensemble* $F = \{f_S\}_{S \in X}$*, does there exist a "global" function* $g \colon V \to \{0,1\}$ *such that* $(\forall S \in X)(f_S = g|_S)$ *where* $g|_S$ *is the restriction of* g *to* S*?*

We can inspect every pair $S, S' \in X$ and certify that $f_S|_{S \cap S'} = f_{S'}|_{S \cap S'}$ which we denote more compactly as $f_S \equiv f_{S'}$. It is not hard to see that the preceding verification succeeds if and only if the ensemble F arises from a global function, i.e., $F = \{f_S = g|_S\}_{S \in X}$ for some $g \colon V \to \{0,1\}$ in which case we say that F is a global ensemble.

We can ask a robust version of this verification procedure, namely, instead of requiring $f_S \equiv f_{S'}$ for **every** pair $S, S' \in X$ we require for **most** pairs. For this reason we introduce a distribution $\mathcal{D}$ on X^2. Now we are ready to formally define an agreement test as follows.

Definition 1.1 (Agreement Test $(X, \mathcal{D})$). *Input: a local ensemble* $F = \{f_S\}_{S \in X}$.

- *Sample* $(S, S') \sim D$*, and*
- *Accept iff* $f_S \equiv f_{S'}$.

We denote the rejecting probability as $\text{disagree}_{(X,D)}(F) := \Pr_{(S,S') \sim D}[f_S \not\equiv f_{S'}]$. Before we define more formally agreement expanders we state a meta-theorem this kind of object should satisfy. An agreement expander is given by a pair $(X, \mathcal{D})$ such the the following meta-theorem holds.

Theorem 1.3 (Meta-Theorem). *Suppose $(X, \mathcal{D})$ is an agreement expander. If* $\mathrm{disagree}_{(X,D)}(F) = \epsilon$, *then there is some $g \colon V \to \{0, 1\}$ such that*

$$\Pr_{S \sim \mathcal{D}} [g|_S \not\equiv f_S] = O(\epsilon),$$

where $S \sim \mathcal{D}$ is the marginal $(S = S, S') \sim \mathcal{D}$.

We will need more notation. Let $\mathrm{Global}(X) \coloneqq \{g \colon V \to \{0, 1\}\}$ be the set of global functions.

Definition 1.2 (Distance to Global). *Given an ensemble $F = \{f_S\}_{S \in X}$, we define the distance* $\mathrm{dist}(F, \mathrm{Global}(X))$ *of F to global as*

$$\mathrm{dist}(F, \mathrm{Global}(X)) \coloneqq \min_{g \in \mathrm{Global}(X)} \Pr_{S \sim \mathcal{D}} [g|_S \not\equiv f_S] .$$

Definition 1.3 (Agreement Expansion). *Given $(X, \mathcal{D})$ we define its agreement expansion as*

$$\mathbb{Y}(X, \mathcal{D}) = \min_{F = \{F_S\}_{S \in X}} \frac{\mathrm{disagree}_{(X,D)}(F)}{\mathrm{dist}(F, \mathrm{Global}(X))},$$

where the minimization is over non-global ensembles.

Definition 1.4 (Agreement Expander). *We say that a family of $\{(X, \mathcal{D})\}$ is an agreement expander family if there exists a constant $\gamma_0 > 0$ such that*

$$\mathbb{Y}(X, \mathcal{D}) \geq \gamma_0,$$

for every member $(X, \mathcal{D})$ of the family.

Note that if $(X, \mathcal{D})$ has agreement expansion $\gamma_0 > 0$ and $F = \{F_S\}_{S \in X}$ is an ensemble satisfying $\mathrm{disagree}_{(X,D)}(F) = \epsilon$ then we conclude that

$$\mathrm{dist}(F, \mathrm{Global}(X)) \leq \frac{\mathrm{disagree}_{(X,D)}(F)}{\gamma_0} = \frac{\epsilon}{\gamma_0},$$

that is F is close to being a global ensemble. Note that this definition shows that agreement expanders indeed satisfy the meta-theorem 1.3.

1.2 High-dimensional Expanders

For us a high-dimensional expander (HDX) is particular kind of hypergraph satisfying some expansion conditions. It is important to stress that there are a few definitions of the notion of high-dimensional expansion. Dinur and Kaufman [1] introduced a new one used in their work which we make precise momentarily.

As a hypergraph a HDX is downward closed in the sense that for every hyperedge all hyperedges contained in it are also part of the hypergraph. This kind of hypergraph is known as simplicial complex.

Definition 1.5 (Simplicial Complex). *We say that a collection X of subsets of $[n]$ is a simplicial complex if for every $S \in X$ and $T \subset S$ we have $T \in X$.*

Hyperedges are usually denoted as faces. Moreover, if $S \in X$ has size k, then we define its dimension to be $k-1$. We say that a non-empty simplicial complex X has dimension d if the largest face has dimension d, i.e., size $d+1$.

Remark 1.4. *The dimension nomenclature comes from the fact that simplicial complexes can arise as geometrical objects. However, for our applications size would be an arguably more natural quantity.*

It is convenient to consider the "slices" $X(i)$ of X according to dimension. More precisely, we set

$$X(i) \coloneqq \{S \in X \mid \dim(S) = i\}.$$

Hence if X is d-dimensional $X = X(-1) \sqcup X(0) \sqcup \cdots \sqcup X(d)$, where $X(-1) = \emptyset$ by convention.

Recall that expander graphs can be seen as (possibly) sparse approximations of complete graphs. A similar phenomenon happens with HDXs where they mimic several properties of a distinguished simplicial complex known as the complete complex.

Definition 1.6 (Complete Complex)**.** *We denote by $\Delta_d(n)$ the collection of all subsets of size $d+1$ of $[n]$ also known as the complete complex of dimension d.*

To talk about the expansion requirements for a simplicial complex to be HDX we need to introduce the standard notion of links.

Definition 1.7 (Link)**.** *Let $S \in X$. We define the link X_S of S as*

$$X_S \coloneqq \{T \setminus S \mid S \subset T \text{ and } T \in X\}.$$

Remark 1.5. *The link X_S is itself a simplicial complex.*

Remark 1.6. *The link $X_\emptyset$ is the entire complex, i.e., $X_\emptyset = X$.*

To any simplicial complex X we can associate a natural graph the so-called 1-skeleton (or simply skeleton).

Definition 1.8 (Skeleton)**.** *The skeleton (or 1-skeleton) of X is the graph $G = (X(0), X(1))$.*

We will work exclusively with a particular kind of simplicial complex called pure.

Definition 1.9 (Pure)**.** *A d-dimensional simplicial complex is pure if for every face $S \in X$ there exists a face $T \in X(d)$ containing S.*

Remark 1.7. *A pure simplicial complex is fully determined by the top slice $X(d)$.*

Let Π_d be a distribution on $X(d)$. Then Π_d gives rise to natural distributions on every $X(i)$. For this we define the following probabilistic experiment. Sample $\mathbf{S}_d \sim \Pi_d$ and for $i = d, \ldots, 1$ do

- Select a uniformly random element $x \in \mathbf{S}_i$, and
- Set $\mathbf{S}_{i-1} = \mathbf{S}_i \setminus \{x\}$.

This process defines a couple array $\vec{\mathbf{S}} = (\mathbf{S}_d, \ldots, \mathbf{S}_0)$ of random variables. It is important to stress that the distribution of $\vec{\mathbf{S}}$ is essential in defining the probabilities and natural random walks associated to HDXs. We define the marginal distribution on $\mathbf{S}_k$ to be Π_k.

Let $S \in X(k)$. We view the skeleton graph $G_S = (V, E)$ of the links X_s as a weighted graph where the weight of $e \in E$ is proportional to $\Pi_{k+2}(e \sqcup S)$.

Now we have all the pieces to define a high-dimensional expander.

Definition 1.10 (γ-HDX)**.** *We say that a d-dimensional simplicial complex X is a γ-HDX provided for every face $S \in X$ of dimension at most $d-2$ the skeleton G_S of X_S is a γ-two-sided spectral expander, i.e., all the eigenvalues of the normalized random walk operator of G_S but the trivial have absolute value at most γ.*

Remark 1.8. *By definition if X is a γ-HDX then the skeleton of X is a γ-two-sided spectral expander.*

Using the distributions Π_k. We define $C^k \coloneqq L^2(X(k))$ space where the inner product is defined as

$$\langle f, g \rangle \coloneqq \mathbb{E}_{S \sim \Pi_k} f(S) g(S),$$

for every $f, g \in C^k$.

Remark 1.9. *It is not hard to show that the complete complex $\Delta_d(n)$ is a γ-HDX with $\gamma = O_d(1/n)$. The issue is that $\Delta_d(n)$ is not sparse, it has $\approx n^d$ hyperedges (assuming d constant). Surprisingly, it is possible to obtain γ-HDXs with arbitrarily small constant γ but only $O_{\gamma,d}(n)$ hyperedges.*

Using the algebraically deep Ramanujan complex constructions of Lubotzky, Vishne and Samuels [5, 4], Dinur and Kaufman showed that sparse γ-HDX do exist.

Theorem 1.10. *For every dimension $d \in \mathbb{N}$ and every expansion $\gamma \in (0, 1)$, there exists an infinite family $X_1, X_2, \ldots$ of explicitly constructible γ-HDXs such that $|X_i| \leq C \cdot |X_i(0)|$ where $C = (1/\gamma)^{O(d^2)}$.*

Open Question 1.11. *Give a combinatorial construction of sparse HDXs (possibly inspired by Zig-Zag product [6]).*

1.3 Natural Random Walks

We define two natural Markov operators. The up operator $U_k \colon C^k \to C^{k+1}$ is defined as

$$(U_k f)(S) \coloneqq \mathbb{E}_{\mathbf{S}_k | \mathbf{S}_{k+1} = S} \left[f(\mathbf{S}_k) \right] = \frac{1}{k+2} \sum_{x \in S} f(S \setminus \{x\}).$$

for every $f \in C^k$ and $S \in X(k+1)$. The down operator $D_k \colon C^k \to C^{k-1}$ is defined as

$$(D_k f)(T) \coloneqq \mathbb{E}_{\mathbf{S}_k | \mathbf{S}_{k-1} = T} \left[f(\mathbf{S}_k) \right],$$

for every $f \in C^k$ and $T \in X(k-1)$.

Remark 1.12. *The constant fuction $1 \in C^k$ is a singular function of both U_k and D_k with singular value 1. Moreover, since U_k and D_k are Markov operators their largest singular value is at most 1.*

Claim 1.13. *U_k and D_{k+1} are adjoint operators.*

Note that U_k and D_{k+1} may be "rectangular" matrices since $|X(k)|$ may differ from $|X(k+1)|$ so the spectral expansion is quantified in terms of singular values.

Definition 1.11 (Spectral Bound)**.** *Let $M\colon L^2(V) \to L^2(W)$ be a Markov operator mapping two finite dimensional vectors spaces V and W. We denote by $\lambda(M)$ be largest non-trivial singular value of M, i.e., largest singular value different from 1.*

Dinur and Kaufman proved that up to γ-error the spectral bound of up and down operators of a γ-HDX match the bound for the complete complex. We defer the proof of the following theorem to 1.5.

Theorem 1.14 (Expansion of Natural Operators)**.** *If X is a γ-HDX, then*

$$\lambda(U_k)^2 = \lambda(D_{k+1})^2 \leq 1 - \frac{1}{k+2} + O(k \cdot \gamma).$$

For our application it will be important that when up (and/or down) operators are composed the resulting operator has arbitrarily good expansion.

Corollary 1.14.1 (Spectral Amplification)**.** $\lambda(U_{k-1} \dots U_{t+1} U_t)^2 \leq \frac{t+1}{k+1} + O((k^2 - t^2) \cdot \gamma)$.

Proof. Indeed

$$\begin{aligned}
\lambda(U_{k-1} \dots U_{t+1} U_t)^2 &\leq \lambda(U_{k-1})^2 \dots \lambda(U_{t+1})^2 \lambda(U_t)^2 \\
&\leq \frac{k}{k+1} \dots \frac{t+2}{t+3} \frac{t+1}{t+2} + O((k^2 - t^2) \cdot \gamma) \\
&\leq \frac{t+1}{k+1} + O((k^2 - t^2) \cdot \gamma).
\end{aligned}$$

□

1.4 Double Samplers

To prove the main theorem of Dinur and Kaufman a possibly weaker object than a HDX suffices, namely, a double sampler. However, currently double samplers are only known to exist as derived objects of HDXs.

Theorem 1.15 (Existence of Double Samplers)**.** *For every $1 < k < d$ and $\epsilon > 0$, there exist an infinite family of tripartite weighted inclusion graphs $G_1, G_2, \dots$ where each G_i has vertex partition $A = [n_i]$, $B = \binom{[n_i]}{k}$ and $C = \binom{[n_i]}{d}$. Furthermore,*

- *$|A \sqcup B \sqcup C| = O_{d,\epsilon}(n_i)$;*
- *For $a \in A, b \in B, c \in C$, $a \sim b$ iff $a \in b$ and $b \sim c$ iff $b \subset c$;* [1]
- *$\lambda(G_i(A, B))^2 \leq \frac{1}{k+2} + \epsilon$; and*
- *$\lambda(G_i(B, C))^2 \leq \frac{k+1}{d+1} + \epsilon$,*

[1] We use $a \sim b$ to denote adjacency of vertices a and b.

where $G_i(A,B)$ and $G_i(B,C)$ are induced bipartite graphs with bipartitions (A,B) and (B,C), respectively.

Proof. Follows directly from the existence of sparse HDXs 1.10 and the spectral amplification 1.14.1. □

Remark 1.16. *For simplicity, we omitted some properties of double samplers used in this application.*

Open Question 1.17. *Construct double samplers without using HDXs.*

1.5 Spectral Bound

We prove the expansion of up and down operators. The proof we present is a slightly simplified argument by Tulsiani of the original proof of Dinur and Kaufman [1].

Theorem 1.18 (Expansion of Natural Operators (restatement))**.** *If X is a γ-HDX, then*

$$\lambda(U_k)^2 = \lambda(D_{k+1})^2 \leq 1 - \frac{1}{k+2} + O(k \cdot \gamma).$$

Proof. It is enough to prove the theorem for D_{k+1} since U_k is its adjoint. Take $f \in C^{k+1}$ such that $\|f\|_2 = 1$, $\langle f, 1 \rangle = 0$ and

$$\lambda \coloneqq \lambda(D_{k+1}) = \frac{\|D_{k+1} f\|_2}{\|f\|_2}.$$

For $i = -1, \ldots, k+1$ set $f_i \coloneqq (D_{i+1} \ldots D_k D_{k+1})\, f$. Clearly, $f_i \in C^i$. Furthermore,

$$0 = \|f_{-1}\|^2 \leq \|f_0\|^2 \leq \cdots \leq \|f_{k+1}\|^2 = 1.$$

Set $\Delta_i = \|f_i\|^2 - \|f_{i-1}\|^2$ so that $\Delta_0 + \Delta_1 + \cdots + \Delta_{k+1} = 1$.

We will need the following claim which crucially uses link expansion.

Claim 1.19.

$$(\forall i \geq -1)(\Delta_{i+2} \geq \Delta_{i+1} - \gamma).$$

We assume the claim and finish the proof of theorem. Then

$$1 = \Delta_0 + \Delta_1 + \cdots + \Delta_{k+1} \leq (k+2) \cdot \Delta_{k+1} + (k+2)^2 \cdot \gamma,$$

implying

$$\Delta_{k+1} \geq \frac{1}{k+2} - (k+2) \cdot \gamma.$$

Alternatively,

$$\Delta_{k+1} = 1 - (\Delta_0 + \cdots + \Delta_k) = 1 - \|f_{k-1}\|^2 = 1 - \left(\frac{\|D_{k+1} f\|_2}{\|f\|_2} \right)^2 = 1 - \lambda^2.$$

Hence

$$\lambda^2 \leq 1 - \frac{1}{k+2} + (k+2) \cdot \gamma,$$

as desired. □

Finally, we prove the claim used in 1.18.

Claim 1.20 (restatement)**.**
$$(\forall i \geq -1)(\Delta_{i+2} \geq \Delta_{i+1} - \gamma).$$

Proof. Let $i \leq d-2$. For every $S \in X(i)$ define
$$h_S(u,v) \coloneqq f_{i+2}(S \sqcup \{u,v\}),$$
where u, v are such that $S \sqcup \{u,v\} \in X(i+2)$. Then
$$\begin{aligned}
\Delta_{i+2} &= \|f_{i+2}\|^2 - \|f_{i+1}\|^2 \\
&= \mathbb{E}_{S\sim\Pi_i}\mathbb{E}_{\{u,v\}\sim X_S(1)} \left(f_{i+2}(S \sqcup \{u,v\})\right)^2 - \mathbb{E}_{S\sim\Pi_i}\mathbb{E}_{u\sim X_S(0)} \left(f_{i+1}(S \sqcup \{u\})\right)^2 \\
&= \mathbb{E}_{S\sim\Pi_i}\mathbb{E}_{\{u,v\}\sim X_S(1)} \left(h_S(u,v)\right)^2 - \mathbb{E}_{S\sim\Pi_i}\mathbb{E}_{u\sim X_S(0)} \left(\mathbb{E}_{v|u} h_S(u,v)\right)^2 \\
&= \mathbb{E}_{S\sim\Pi_i}\mathbb{E}_{\{u,v\}\sim X_S(1)} \left(h_S(u,v) - \mathbb{E}_{v'|u} h_S(u,v')\right)^2.
\end{aligned}$$

Similarly,
$$\Delta_{i+1} = \mathbb{E}_{S\sim\Pi_i}\mathbb{E}_{u\sim X_S(0)} \left(\mathbb{E}_{v|u} h_S(u,v) - \mathbb{E}_{\{u,v\}\sim X_s(1)} h_S(u,v)\right)^2.$$
Let $G_S = (V,E)$ be the weighted graph of the 1-skeleton of the link X_s. Let $\mu_S \coloneqq \mathbb{E}_{\{u,v\}\sim X_S(1)} h_S(u,v)$. Define
$$h_2(u,v) \coloneqq h_S(u,v) - \mu_S,$$
and
$$h_1(u) \coloneqq \mathbb{E}_{v|u} h_S(u,v) - \mu_S.$$
Observe that $\mathbb{E}_{u\sim X_s(0)} h_1(u) = 0$. Let A_S be the normalized random walk operator of G_S. Then
$$\begin{aligned}
\mathbb{E}_{\{u,v\}\sim X_S(1)} \left(h_2(u,v) - h_1(u)\right)^2 &\geq \mathbb{E}_{v\sim X_s(0)} \left(\mathbb{E}_{u|v} h_2(u,v) - \mathbb{E}_{u|v} h_1(u)\right)^2 && \text{(Jensen's Inequality)} \\
&= \mathbb{E}_{v\sim X_s(0)} \left(h_1(v) - \mathbb{E}_{u|v} h_1(v)\right)^2 \\
&= \|(I - A_S)h_1\|^2 \\
&\geq (1-\gamma)\|h_1\|^2. && \text{(γ-two-sided spectral expander)}
\end{aligned}$$
Combining the above expressions gives $\Delta_{i+2} \geq (1-\gamma) \cdot \Delta_{i+1}$. □

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