Boolean function analysis on high-dimensional expanders

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From classical Fourier analysis, any function on the Boolean cube $\{0,1\}^n$ admits an orthogonal decomposition into homogeneous parts, and this also extends to the d-dimensional complete complex $\Delta_d(n)$, which is the set of all subsets of $[n]$ of size at most $d+1$, as long as $d+1 \leq \frac{n}{2}$. A d-dimensional high-dimensional expander is often viewed and used as a sparse approximation of $\Delta_d(n)$, and this work shows that indeed a function on the hyperedges of an HDX also admits an (approximate) orthogonal decomposition. We first define a condition called *properness* that is sufficient for a simplicial complex to admit such decompositions.

Recall that a simplicial complex is simply a downward complex hypergraph. A simplicial complex is pure if for every subset $t \in X$, there is some superset of t in the top layer $X(d)$.

1 When do functions decompose?

We use the same definitions and notations as in previous weeks. In particular, we focus on the linear algebraic properties of the operators $U_i : C^i \to C^{i+1}$ and $D_{i+1} : C^{i+1} \to C^i$, and observe that U_i and D_{i+1} are adjoints under the inner product $\langle f, g\rangle = \mathbb{E}_{s\in\Pi_i}[f(s)g(s)]$, where $f, g \in C^i$. We wish to decompose the space C^k.

Let $H^i \subseteq C^i$ be the space of all functions that are killed by the D_i operator, that is, $H_i := \ker D_i$, and let V^i be the *lifting* of H^i to C^k by the Up walks. That is, $V^i := U^{k-i}H^i$, where U^{k-i} is a shorthand for $U_{k-1}\cdots U_{i+1}U_i$, used when the domain of concantenated up walk is clear. Since D_{-1} does not exist, we let $H_{-1} := C^{-1} = \mathbb{R}$.

Definition 1.1. *A k-dimensional simplicial complex is* proper *if the operator $D_{i+1}U_i$ is positive definite for all $-1 \leq i \leq k-1$. Because $\langle D_{i+1}U_i f, f\rangle = \langle U_i f, U_i f\rangle = ||U_i f||^2$, this is equivalent to* $\ker(U_i)$ *being trivial, or U_i being full-rank/injective.*

Theorem 1.2 (Decomposition Theorem)**.** *If X is proper, then $C^k = V^k \oplus V^{k-1} \oplus V^{k-2} \oplus \cdots \oplus V^{-1}$.*

Proof. Induction on k. Base case (k=-1) is true as $V^{-1} = H^{-1} = C^{-1}$.

Let $f \in C^k$. We have, $C^k = \ker D_k \oplus \operatorname{im} D_k^* = \ker D_k \oplus \operatorname{im} U_{k-1}$, and so $f = h_k + h$, where $h_k \in \ker D_k = V^k$ and $h \in \operatorname{im} U_{k-1}$. By properness, U_{k-1} is an injective map, so we can find a *unique* $g \in C^{k-1}$ such that $h = U_{k-1}g$. By induction hypothesis, g can be uniquely decomposed into $g = h_{k-1} + U_{k-2}h_{k-2} + U_{k-2}U_{k-3}h_{k-3} + \cdots + (U_{k-2}U_{k-3}\cdots U_{-1})h_{-1}$, with $h_i \in H^i$, and this means $f = h_k + Uh_{k-1} + U^2 h_{k-2} + \cdots + U^{k+1}h_{-1}$, if we use the concatenated walk notation.

Given f, the choice for h_i is unique throughout the inductive step, and so $f = f_{-1} + f_0 + \cdots + f_k$, where $f_i \in V^i$, and this decomposition is unique. □

The analogous decomposition over the Boolean hypercube is into homogeneous functions. It turns out we can define a similar notion of homegenity here.

Proposition 1.3 (Analog of monomial)**.** *Given a face $s \in X(i)$, y_s is the indicator of containing the face s. y_s is a function in C^j for all $j \geq i$. Moreover, the functions y_s across the levels are related as $U_j y_s = (1 - \frac{i+1}{j+2}) y_s$, where the y_s on LHS is in C^j, and the one on RHS is in C^{j+1}.*

Proof. $(U_j y_s)(t) = \frac{1}{|t|} \sum\limits_{x \in t} y_s(t \backslash \{x\}) = \frac{|t|-|s|}{|t|} y_s(t)$ □

Lemma 1.4 (Homogeneous functions)**.** *Let X be a proper simplicial complex, and the spaces be defined as above. Every function $h \in V^i$ can be uniquely represented as $h = \sum\limits_{s \in X(i)} \tilde{h}(s) y_s$. Because all s that appear in the decomposition have the same dimension, h are analogous to homogeneous polynomials. The coefficients also satisfy the following* harmonicity *condition:*

$$\sum_{s \supset t} \Pi_i(s) \tilde{h}(s) = 0$$

Proof. Let $h = U^{k-i} g$, where $g \in H^i$.

$$\begin{aligned} g &= \sum_{s \in X(i)} g(s) y_s \\ h = U^{k-i} g &= \sum_{s \in X(i)} g(s) U^{k-i} y_s \\ &= \sum_{s \in X(i)} g(s)(1 - \frac{i+1}{i+2})(1 - \frac{i+1}{i+3}) \cdots (1 - \frac{i+1}{k+1}) y_s \\ &= \sum_{s \in X(i)} \tilde{h}(s) y_s \end{aligned}$$

The harmonicity condition follows from the fact that $g \in H^i \implies D_i g = 0$.

Clearly, the coefficients $g(s)$ are unique for a given g. So as long as the lifting U^{k-i} is injective (for a unique g corresponding to h), we have a unique representation, and this is exactly what *proper* complexes offer. □

Combining Lemma 1.4 and Theorem 1.2, we see that any function $f \in C^k$ can be uniquely written as $f = \sum_{s \in X} \tilde{h}(s) y_s$.

Definition 1.5. *The largest d such that there is some face $s \in X(d-1)$ with $\tilde{h}(s) \neq 0$ is defined to be the degree of f. This is completely analogous to the usual decomposition, where degree of a function is the degree of the largest monomial that appears in its decomposition.*

It follows that $\{y_s\}_{s \in X(-1) \cup X(0) \cup \cdots \cup X(d)}$ spans the set of all functions of degree at most $d+1$. Is there linear dependence in this spanning set? Yes, and the following lemma finds a basis for functions of degree at most $d+1$. It is important to note that the functions themselves still reside in C^k, only their degree is at most $d+1$.

Lemma 1.6. *The set $\{y_s\}_{s \in X(d)}$ is a basis for the space of functions of degree at most d.*

Proof. If some set $r \in X(k)$ contains t, where $|t| \le d$, then it contains $\binom{k+1-|t|}{d+1-|t|}$ many supersets of t in $X(d)$. This means, $y_t = \frac{1}{\binom{k+1-|t|}{d+1-|t|}} \sum\limits_{t \subset s, s \in X(d)} y_s$.

Now we need to show that $\{y_s\}_{s \in X(d)}$ has same number of elements as the dimension of at most degree $d+1$ functions, which is $\sum_{i=-1}^{d} \dim V^i = \sum_{i=-1}^{d} \dim H^i = \sum_{i=-1}^{d} (\dim C^i - \dim C^{i-1}) = \dim C^d = |X(d)|$. This completes the proof as $|\{y_s\}_{s \in X(d)}| = |X(d)|$. □

It turns out that with the right definition of high dimensional expansion, it is easy to ensure properness. Moreover, under this high dimensional expansion, the decomposition also has other nice properties. Before we see what buys us properness in Section 2, let us see an analog of the classic Degree-1 Theorem from Boolean Function Analysis on proper simplicial complexes.

Theorem 1.7 (Degree-1 Theorem)**.** *Suppose X is a proper k-dimensional simplicial complex with $k \ge 2$, whose 1-skeleton is connected. A function $f \in C^k$ is Boolean and degree-1 iff f is the indicator of intersecting I or of not intersecting I for some independent set I of the 1-skeleton.*

Proof. Let $f = \sum_{v \in X(0)} c_v y_v$, so $0 = f^2 - f = \sum_{v \in X(0)} (c_v^2 - c_v) y_v + \sum_{\{u,v\} \in X(1)} c_u c_v y_{\{u,v\}}$.

Substituting $y_v = \frac{1}{\binom{k+1-1}{1+1-1}} \sum\limits_{u:\{u,v\} \in X(1)} y_{\{u,v\}} = \frac{1}{k} \sum\limits_{u:\{u,v\} \in X(1)} y_{\{u,v\}}$,
and using the fact that $\{y_{\{u,v\}}\}_{\{u,v\} \in X(1)}$ is a basis, we conclude all coefficients must be zero.

This means, $2kc_uc_v = c_u(1-c_u) + c_v(1-c_v)$ for any $\{u,v\} \in X(1)$. In fact, for any $\{u,v,w\} \in X(2)$,

$$
\begin{aligned}
2kc_uc_v &= c_u(1-c_u) + c_v(1-c_v) \\
2kc_uc_w &= c_u(1-c_u) + c_w(1-c_w) \\
2kc_wc_v &= c_w(1-c_w) + c_v(1-c_v)
\end{aligned}
$$

Solving this system, we see that two cases are possible:

1. Two of c_u, c_v, c_w are 0, and the third is 0 or 1.
2. Two of c_u, c_v, c_w are $\frac{1}{k+1}$, and the third is $\frac{1}{k+1}$ or $1 - \frac{1}{k+1}$.

There cannot be an edge between u, v if u and v do not belong to the same case (consider triangle containing $\{u,v\}$), and because graph is connected, all vertices belong to one of the two cases. In the first case, all vertices u for which $c_u = 1$ must form an independent set. So, f is the function that determines intersection with the independent set I, where $I = \{u : c_u = 1\}$.

If f corresponds to second case, then $1-f$ corresponds to the first case, because $1 = \frac{1}{k+1} \sum\limits_{u \in X(0)} y_u$, and so it is the indicator of not intersecting the independent set $I = \{u : c_u = 1 - \frac{1}{k+1}\}$. □

2 Chasing Properness

The operators $U_{i-1}D_i$ and $D_{i+1}U_i$ can be seen as random walks on $X(i)$, which we call lower random walk and upper random walk respectively, and represent as UD and DU when the level is clear from context. Given current $s \in X(i)$, the probability that the next face under random walk

UD (or DU) is $t \in X(i)$ is given by the weight of $f(t)$ while evaluating $UDf(s)$ (or $DUf(s)$). Let M_i^+ be the walk DU but constrained to be non-lazy, that is,

$$DU = \frac{1}{i+2} I + \frac{i+1}{i+2} M_i^+$$

If X were a d-regular unweighted expander, then X would have dimension 1, and Π_0 and Π_1 would both be uniform. In this case, M_0^+ corresponds to the standard adjacency walk, while UD is the walk that chooses a vertex uniformly at random at every step. The statement that second eigenvalue of this expander is λ would be equivalent $|M_0^+ - UD| < \lambda$. Generalizing this notion, we have the following:

Definition 2.1. *Let X be a k-dimensional simplicial complex, with probability distribution Π_k on the top face from which Π_i is derived as before, and let DU, UD and M_i^+ be walks on $X(i)$. For a $\gamma < 1$, we say X is a γ-high-dimensional-expander (or γ-HDX) if for every $0 \leq i \leq k-1$,*

$$|M_i^+ - UD| < \gamma$$

where the norm used is operator norm: $|A| = \max_{|x|=1} \langle Ax, x \rangle$.

This definition easily yields properness, as the following lemma shows:

Lemma 2.2. *If X is a k-dimensional γ-HDX with $\gamma < \frac{1}{k+1}$, then X is proper.*

Proof. Let $f \in C^j$.

$$\begin{aligned}
\langle Uf, Uf \rangle = \langle DUf, f \rangle &= \frac{1}{j+2}\langle f, f \rangle + \frac{j+1}{j+2}\langle M_j^+ f, f \rangle \\
&= \frac{1}{j+2}\langle f, f \rangle + \frac{j+1}{j+2}(\langle (M_j^+ - UD)f, f \rangle + \langle UDf, f \rangle) \\
&\geq \frac{1}{j+2}||f||^2 - \frac{j+1}{j+2}||M_j^+ - UD||\ ||f||^2 + \frac{j+1}{j+2}\langle Df, Df \rangle \\
&\geq \frac{1}{j+2}||f||^2 - \frac{j+1}{j+2}\gamma ||f||^2 + 0 \\
&= ||f||^2 (\frac{1}{j+2} - \frac{j+1}{j+2}\gamma)
\end{aligned}$$

The last term is strictly positive if $\gamma < \frac{1}{j+1}$, and so $\gamma < \frac{1}{k+1}$ is sufficient. □

But hadn't we defined a different definition for HDX in previous weeks based on [2] in terms of spectral link expansion? The two definitions generalize different properties of expanders, and we use this definition here as this readily yields properness. Fortunately, however, the two definitions turn out to be equivalent as Section 5 of [1] shows. We skip the proof of equivalence for the sake of brevity. In this document, we work only with this new definition, and this is the definition that the term γ-HDX will refer to.

One of the classic results in Fourier decomposition is the Friedgut-Kalai-Naor (FKN) Theorem, which can be seen as the robust version of Degree 1 Theorem, and we work towards proving an FKN theorem for γ-HDX in the next section.

3 FKN Theorem on High Dimensional Expanders

In [3], the author proves FKN Theorem for the Boolean slice, as the following theorem shows.

Theorem 3.1 (FKN Theorem on the slice [3])**.** *Let $n, k \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in (0,1)$ such that $n/4 \leq k+1 \leq n/2$. Let $F : \binom{[n]}{k+1} \to \{0,1\}$ be a Boolean function such that $\mathbb{E}[(F-f)^2] < \epsilon$ for some degree 1 function $f : \binom{[n]}{k+1} \to \{0,1\}$. Then there exists a degree 1 function $g : \binom{[n]}{k+1} \to \mathbb{R}$ such that*

$$\Pr[F \neq g] = O(\epsilon)$$

Furthermore, $g \in \{0, 1, y_i, 1-y_i\}$, that is, g is a Boolean dictator.

We are looking for an analog of the above theorem for High Dimensional Expanders. The proof follows a generic strategy to extend theorems from Boolean slice to High Dimensional Expanders. The idea is that if we wish to prove the FKN Theorem for level k, we move to a higher level (in this case $2k, 4k$ levels), and fix a set t on the higher level. Now because of downward closure, all subsets of t of level k will be present, and now we can apply the theorem for slice (in this case Theorem 3.1). Now the task reduces to stitch together the solutions obtained for different views from different t. In this case, we take help of Agreement Theorem from [2].

In [2], the authors prove an agreement theorem given below. $\mathcal{D}_{k,2k}$ is the ditribution on pairs of elements from $X(k)$ obtained by first sampling an element t of $X(2k)$, and then sampling two subsets in $X(k)$ of t uniformly at random.

Theorem 3.2 (Agreement Theorem for High Dimensional Expanders)**.** *Let X be a d-dimensional $\lambda < \frac{1}{3d^2}$, let $k^2 < d$, and let Σ be some fixed alphabet. Let $\{f_s : s \to \Sigma\}_{s \in X(k)}$ be an ensemble of local functions on $X(k)$, i.e. $f_s \in \Sigma^s$ for each $s \in X(k)$. If*

$$\Pr_{(s_1,s_2)\in\mathcal{D}_{k,2k}}[f_{s_1}|_{s_1\cap s_2} \equiv f_{s_2}|_{s_1\cap s_2}] > 1-\epsilon$$

then there is a $g : X(0) \to \Sigma$ such that

$$\Pr_{s\sim\Pi_k}[f_s \equiv g|_s] \geq 1 - O_\lambda(\epsilon)$$

Below we give a rough sketch of how the FKN Theorem may be extended to High Dimensional Expanders. Please look at the paper for details. We borrow terminology from Theorem 3.1.

Fix a $t \in X(2k)$ and let ϵ_t be the agreement between F and f on the part of $X(k)$ that has subsets of t. $\mathbb{E}_{t\in X(2k)}[\epsilon_t] = \epsilon$. Focusing on these restrictions $f|_t$ and $F|_t$, we can apply Theorem 3.1 to get Boolean dictators $g_t : \binom{t}{k} \to \{0,1\}$ that have agreement $O(\epsilon_t)$ to $f|_t$ (and $F|_t$). g_t can be written as a string in Σ^t by taking the degree 1 representation $g_t = \sum_{i\in t} d_t(i) y_i$, where $d_t : t \to \{0, 1, \frac{1}{k+1}, \frac{k}{k+1} = \Sigma$, and d_t can been as a string in Σ^t. To be able to apply Theorem 3.2 how can we ensure that d_t strings agree on their intersections? That is, we need to bound $\Pr_{(t_1,t_2)\in\mathcal{D}_{2k,4k}}[d_{t_1}|_{t_1\cap t_2} \not\equiv d_{t_2}|_{t_1\cap t_2}]$.

This is where the $4k$-dimension layer comes in. Just as there are strings $d_t \in \Sigma^t$ for $t \in X(2k)$, there are also strings $e_u \in \Sigma^u$ for $u \in X(4k)$ (and boolean dictators h_u corresponding to g_t). If we can say that for $t \subset u$, d_t string must agree with the restriction of e_u to t, then we are done, because then d_{t_1} and d_{t_2} cannot be different either.

If $t \subset u$ and d_t differs from (restriction of) e_u, then their corresponding functions g_t and $h_u|_t$ are also different. But if two Boolean dictators differ, their disagreement is $\Omega(1)$. Moreover, g_t is $O(\epsilon)$ close to $f|_t$ on average, and h_u is $O(\epsilon)$ close to $f|_u$ on average, and so by triangle inequality $\mathbb{E}_u \mathbb{E}_{t \subset u} \mathbb{E}[||g_t - h_u|_t] \approx \mathbb{E}_u \mathbb{E}_{t \subset u} \mathbb{E}[(f|_t - f_u|_t)^2] = 0$. This means that d_t and e_u cannot differ too often, and as a consequence, $\Pr_{(t_1,t_2) \in \mathcal{D}_{2k,4k}} [d_{t_1}|_{t_1 \cap t_2} \not\equiv d_{t_2}|_{t_1 \cap t_2}]$ is upper bounded to be $O(\epsilon)$.

Therefore, we can use Theorem 3.2 and get a global string $d : X(0) \to \Sigma$, which corresponds to a function $g = \sum_{i \in X(0)} d(i) y_i$ that agrees with g_t locally (so is approximately Boolean), and hence with f (and F) as well.

4 Generalizing high dimensional expansion to Posets: Eposets

In this section, we generalize γ-HDX to general posets, and derive interesting properties of the decomposition in the next section. To generalize the notion fully, we need analogs of simplicial complexes, purity, properness and γ-HDX (Definition 2.1).

First, recall that informally speaking, a ranked or graded poset is a poset that has some notion of rank in $\mathbb{Z}$ associated with every element that respects the partial order. For sets, this is the size of set, and for subspaces this is the dimension (strictly speaking, in this case size and dimension are 1+rank). We will be dealing k-dimensional posets, where the rank of all elements is between -1 and k. A k-dimensional poset is pure if every element of rank $< k$ has some element on the level k (elements with rank k) that is greater under the partial order.

Definition 4.1 (Analog of simplicial complex)**.** *Let X be a finite ranked pure d-dimensional poset with a unique minimum element of rank -1. Denote by $X(i)$ all elements in X of rank i. We say X is* measured *by a distribution $(\Pi_d, \Pi_{d-1}, \cdots, \Pi_{-1})$ if it satisfies the following properties:*

1. *Π_i is a distribution on $X(i)$*
2. *$\Pi_{i-1} \subset \Pi_i$ for $i > -1$*
3. *Π_{i-1} depends only on Π_i for $i > -1$*

Given these, we can again define the function spaces C^j on $X(j)$, and the walk operators U_j, D_{j+1}. It is easy to see the complete complex $\Delta_d(n)$ and Grassmann poset $\mathrm{Gr}_q(n,d)$ are both covered by this definition. The definition of properness is also easily extended now, and we say a measured poset is *proper* if $\ker U_j$ is trivial for all $j \le d$.

Finally, we complete the last leg of this generalization, getting γ-eposet as an analog of γ-HDX.

Definition 4.2 (γ-eposet)**.** *Let $\overrightarrow{r}, \overrightarrow{\delta} \in \mathbb{R}^k_{\ge 0}$, and let $\gamma < 1$. A measured poset X is a $(\overrightarrow{r}, \overrightarrow{\delta}, \gamma)$-eposet if for $0 \le j \le k-1$,*

$$||D_{j+1}U_j - r_j I - \delta_j U_{j-1} D_j|| \le \gamma$$

As should be clear, we can fit in γ^*-HDX into this definition by letting $r_j = \frac{1}{j+2}, \delta_j = 1 - \frac{1}{j+2}$ and $\gamma = \gamma^*/2$ because then

$$||D_{j+1}U_j - \frac{1}{j+2} I - \frac{j+1}{j+2} U_{j-1} D_j|| \le \frac{1}{2}\gamma^* \le \frac{j+1}{j+2}\gamma^*$$
$$||M_j^+ - UD|| \le \gamma^*$$

It can be shown that any γ-eposet admits decompositions similar to those in Section 1. Moreover, we can now show that the decomposition is approximately orthogonal and approximately an eigendecomposition.

5 Properties of decomposition in Eposets

Theorem 5.1 (Properties of decomposition). *Let X be a k-dimensional $(\overrightarrow{r}, \overrightarrow{\delta}, \gamma)$-eposet. For every function $f \in C^l (l \leq k)$, the decomposition $f = f_{-1} + f_0 + \cdots + f_l$ satisfies the following properties for small enough γ:*

1. *(Orthogonality) For $i \neq j, |\langle f_i, f_j \rangle| = O(\gamma)||f_i||\ ||f_j||$.*
2. *$||f||^2 = (1 + \pm O(\gamma))(||f_{-1}||^2 + ||f_0||^2 + \cdots + ||f_l||^2)$.*
3. *(Eigendecomposition) If $l < k$, f_i are approximate eigenvectors of DU operator: $||DUf_i - r^l_{l-i+1} f_i|| = O(\gamma)||f_i||$.*
4. *If $l < k$, $\langle DUf, f \rangle = \sum_{i=-1}^{l} r^l_{l-i+1} ||f_i||^2 \pm O(\gamma)||f||^2$*

Remark 5.2. *A proper measured poset is sufficient for decomposition to exist. But we require it to be an eposet for the decomposition to have these properties.*

Remark 5.3. *The last condition implies that X is proper for $\overrightarrow{r} > 0$ and small enough γ, and so the decomposition exists.*

Remark 5.4. *An important thing to note is that the constants hidden in asymptotics depend only on $k, \overrightarrow{r}, \overrightarrow{\delta}$, and not on $|X|$. In the known constructions of HDX, the blowup $\frac{|X|}{|X(0)|}$ increases rapidly as the expansion γ decreases (as $(\frac{1}{\gamma})^{O(k^2)}$), and so it is important that the error terms are independent of $|X|$, or decreasing γ will increase the error via $|X|$. See [2] for more details.*

Item 2 follows readily from item 1, and item 4 follows from item 3. So we focus on proving items 1 and 3.

5.1 Proof of Item 3

With the right indices, we know that $DU \approx rI + \delta UD$ with the error bounded by γ. Now, $DU^2 = (DU)U \approx (rI + \delta UD)U = rU + \delta U(DU) \approx rU + \delta U(r'I + \delta' UD) = (r + \delta r')U + \delta\delta' U^2 D$, and the error is bounded by $\gamma||U|| + \delta\gamma||U|| \leq (1 + \delta)\gamma = O(\gamma)$ because $||U|| \leq 1$.

We observe that even if we repeat these steps multiple times starting from DU^j instead of DU^2, the error term will always be bounded as $O(\gamma)$ as long as $j \leq k$ (we treat $k, \overrightarrow{r}, \overrightarrow{\delta}$ as constants). These calculations may be turned into a formal inductive proof that for $1 \leq j \leq l + 1 \leq k$, $DU^j : X(l - j + 1) \to X(l)$,

$$||DU^j - r^l_j U^{j-1} - \delta^l_j U^j D|| = O(\gamma) \tag{1}$$

where $\delta^l_j = \delta_l \delta_{l-1} \cdots \delta_{l-j+1}$, and
$r^l_j = r_l + r_{l-1}\delta_l + r_{l-2}\delta_l\delta_{l-1} + \cdots + r_{l-j+1}(\delta_l\delta_{l-1}..\delta_{l-j+2}) = r_l + r_{l-1}\delta^l_1 + r_{l-2}\delta^l_2 + \cdots r_{l-j+1}\delta^l_{j-1}$.

Recall that in the decomposition, $\exists h_i \in H^i$ such that $U^{l-i}h_i = f_i$. Applying Equation 1 to h_{l-j+1}, and using $H^{l-j+1} = \ker D_{l-j+1}$, we conclude that

$$||DUf_{l-j+1} - r_j^l f_{l-j+1} - 0|| = ||DU^j h_{l-j+1} - r_j^l U^{j-1} h_{l-j+1} - \delta_j^l U^j D h_{l-j+1}|| = O(\gamma)||f_{l-j+1}||$$

Replacing $l-j+1$ by i, we get $||DUf_i - r_{l-i+1}^l f_i|| = O(\gamma)||f_i||$, proving item 3.

5.2 Proof of Item 1

There are two parts to this proof. First, we show $\langle f_i, f_j\rangle = O(\gamma)||h_i||\ ||h_j||$, and then we show that $||f_i|| = (1 \pm O(\gamma)) r_{l-i}^l r_{l-i-1}^{l-1} \cdots r_0^i ||h_i||$. Clearly, the two together prove item 1.

(i) For $i \neq j$, $\langle f_i, f_j\rangle = O(\gamma)||h_i||\ ||h_j||$

Proof by induction on l. Base case $l = 0$ means that either $i = l$ or $j = l$ (not both). WLOG let $i = l$, then $\langle f_i, f_j\rangle = \langle h_l, U^{l-j}h_j\rangle = \langle D_l h_l, U^{l-j-1}h_j\rangle = 0$ because $h_l \in \ker D_l$.

Now suppose the claim holds till $l-1$. $\langle f_i, f_j\rangle = \langle U^{l-i}h_i, U^{l-j}h_j\rangle = \langle DU^{l-i}h_i, U^{l-j-1}h_j\rangle$. Using equation 1 on h_i, we can see that $DU^{l-i}h_i = U^{l-i-1}h_i + U^{l-i}Dh_i + \Gamma_i$, where Γ_i is the error term bounded by $O(\gamma)||h_i||$. But $U^{l-i}Dh_i = 0$, and so,

$$\begin{aligned}
\langle f_i, f_j\rangle &= \langle U^{l-i-1}h_i + \Gamma_i, U^{l-j-1}h_j\rangle \\
&= \langle U^{l-i-1}h_i, U^{l-j-1}h_j\rangle + \langle \Gamma_i, U^{l-j-1}h_j\rangle \\
&\leq O(\gamma)||h_i||\ ||h_j|| + ||\Gamma_i||\ ||U^{l-j-1}||\ ||h_j|| \\
&\leq O(\gamma)||h_i||\ ||h_j|| + O(\gamma)||h_i||\ ||h_j|| = O(\gamma)||h_i||\ ||h_j||
\end{aligned}$$

(ii) $||f_i|| = (1 \pm O(\gamma)) r_{l-i}^l r_{l-i-1}^{l-1} \cdots r_0^i ||h_i||$. Once again, using equation 1,

$$\begin{aligned}
||f_i||^2 = \langle f_i, f_i\rangle &= \langle U^{l-i}h_i, U^{l-i}h_i\rangle \\
&= \langle D^{l-i}U^{l-i}h_i, h_i\rangle \\
&= \langle D^{l-i-1}DU^{l-i}h_i, h_i\rangle \\
&= \langle D^{l-i-1}U^{l-i-1}h_i + D^{l-i-1}\Gamma_i, h_i\rangle \\
&= \langle D^{l-i-1}U^{l-i-1}h_i, h_i\rangle + \langle D^{l-i-1}\Gamma_i, h_i\rangle \\
&\leq \langle D^{l-i-1}U^{l-i-1}h_i, h_i\rangle + ||D^{l-i-1}\Gamma_i||\ ||h_i|| \\
&\leq \langle D^{l-i-1}U^{l-i-1}h_i, h_i\rangle + ||\Gamma_i||\ ||h_i|| \\
&\leq \langle D^{l-i-1}U^{l-i-1}h_i, h_i\rangle + O(\gamma)||h_i||^2
\end{aligned}$$

And this calculation can be extended to obtain $||f_i||^2 = ||h_i||^2 + O(\gamma)||h_i||^2$.

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