Expander Graphs - Both Local and Global

October 15, 2019

1 Motivation

These notes are based on Chapman presentation on his joint work with Linial and Peled [1].

- Random d-regular graphs are good expanders but bad HDXs
- Bounded degree HDX come from group theory [1]
- Goal: construct bounded degree HDXs with more straightforward combinatorial ideas and satisfying some regularity conditions.

Definition 1.1. *We call a graph $G = (V, E)$ (a, b)-regular if G is a-regular and for all $v \in V$, G_v (the neighborhood of v) is b-regular.*

Some examples of (a, b)-regular construction are:

- Cliques are $(d-1, d-2)$-regular.
- k-regular nice enough triangulation of surfaces are $(k, 2)$-regular.
- LSV Ramanujan Complexes of type $\tilde{A}_2$ are $(2q^2 + 2q + 2, q + 1)$-regular.
- Kaufman-Oppenheim construction.
- Polygraphs (this work).

Question 1.1. *Fixing (a, b). Are there interesting infinite families of (a, b)-expanders?*

2 Polygraphs

Definition 2.1 (Polygraphs)**.** *Given a multi-set $S = [\ell_1, \dots, \ell_m]$ of non-negative integers and a d-regular graph G with $\mathrm{girth}(G) > 3\max(S)$, then the polygraph G_S is defined as follows $V_{G_S} = V_G^m$ and $(v_1, \dots, v_m) \sim (u_1, \dots, u_m)$ if $[\rho(u_1, v_1), \dots, \rho(u_m, v_m)] = S$ where $\rho \colon V \times V \to \mathbb{R}^+$ is a metric on G (for this exposition think of ρ as the usual distance metric).*

We illustrate the definition with some well known graph products. Note that the usual cartesian product $G \square G$ is the polygraph G_S where $S = [1, 0]$. Similarly, the tensor product $G \otimes G$ is the polygraph G_S where $S = [1, 1]$.

[1] Now, we also have Liu et al. combinatorial construction.

Remark 2.1. *If G is (a,b)-regular and H is (c,d)-regular, then $G \otimes H$ is (ac, bd)-regular.*

Claim 2.2. *If G is a d-regular graph with high enough girth then G_S is (a,b)-regular where $S = [\ell_1, \dots, \ell_m]$ with possibly $b = 0$.*

Proof. Note that a can be computed explicitly. Moreover, from the high girth of G (locally it looks like a tree) we have a transitive action on neighborhoods establishing b-regularity. $\square$

Now, we investigate the polygraphs with $S = [1,1,0]$. Let A_G be the normalized adjacency operator of G with second largest eigenvalue λ. Then

$$A_{G_{[1,1,0]}} = A_G \otimes A_G \otimes I + A_G \otimes I \otimes A_G + I \otimes A_G \otimes A_G,$$

has second largest eigenvalue bounded by $3\lambda^2$.

3 Open Problems

Conjecture 3.1. *The local expansion of G_S does not depend at all on d.*

	Discrepancy	Geometric Overlap	Coboundary Expansion	Mixing Edge-Δ-Edge
$[1,1,0]$	No	?	No	Yes
$[1,2,3]$	?	Yes	?	Yes

References

[1] Michael Chapman, Nati Linial, and Yuval Peled. Expander graphs – both local and global. *CoRR*, abs/1812.11558, 2018.