Oppenheim's Trickling-Down Theorem

October 12, 2019

1 Oppenheim's Trickling-Down Theorem

A convenient way of certifying that a simplicial complex is a HDX is to show that the links of co-dimension 2 are expanding and all other links of co-dimension 3 or higher are connected. This sufficient condition for high-dimensional expansion was established by Oppenheim [1] and is now commonly referred as Oppenheim's trickling-down Theorem. Oppenheim's result is similar in spirit to the so-called Garland's method and it is a manifestation of a local to global phenomenon. Here, we follow the exposition of Yotam Dikstein from the HDX cluster at Simons.

Towards proving this sufficient condition a key result is the particular case of a simplicial complex of dimension 2.

Theorem 1.1 (dimension two)**.** *Suppose X is a simplicial complex of dimension 2 endowed with a measure Π_2 on the top facets. If*

(i) the second largest eigenvalue of every link X_v for $v \in X(0)$ is bounded by λ, and

(ii) the underlying skeleton of X is connected,

then the skeleton has second largest eigenvalue bounded by $\lambda/(1-\lambda)$.

We will need two simple results along the way and a bit of notation. For a simplicial complex X and $f \colon X(0) \to \mathbb{R}$, we denote the restriction $f|_{X_v(0)}$ by f_v. Also, we denote by A_v the adjacency operator of the link of $v \in X(0)$. The first result is "localization", i.e., informally speaking it captures local behavior at the links.

Claim 1.2 (localization)**.** *Let $f, g \colon X(0) \to \mathbb{R}$. Then*

(i) $\langle f, g\rangle = \mathbb{E}_{v \sim X(0)} f(v)g(v) = \mathbb{E}_{v \sim X(0)} \mathbb{E}_{u \sim X_v(0)} f(u)g(u)$,

(ii) $\langle Af, g\rangle = \mathbb{E}_{v \sim X(0)} \langle A_v f_v, g_v\rangle$.

Proof. We start with the first item. Note that sampling a vertex $v \sim X(0)$ can be done equivalently by sampling an edge $uv \sim X(1)$ and then taking u or v with equal probability. Hence

$$\begin{aligned}
\mathbb{E}_{v\sim X(0)} f(v)g(v) &= \sum_{uv\in X(1)} f(u)g(u)\frac{1}{2}\Pr[uv] + f(v)g(v)\frac{1}{2}\Pr[uv] \\
&= \sum_{v\in X(0)} \sum_{u\in X_v(0)} f(u)g(u)\frac{1}{2}\Pr[uv] + \sum_{u\in X(0)} \sum_{v\in X_v(0)} f(v)g(v)\frac{1}{2}\Pr[uv] \\
&= \frac{1}{2}\mathbb{E}_{v\sim X(0)} \sum_{u\in X_v(0)} f(u)g(u)\Pr[uv|v] + \frac{1}{2}\mathbb{E}_{u\sim X(0)} \sum_{v\in X_u(0)} f(v)g(v)\Pr[uv|u] \\
&= \frac{1}{2}\mathbb{E}_{v\sim X(0)}\mathbb{E}_{u\sim X_v(0)} f(u)g(u) + \frac{1}{2}\mathbb{E}_{u\sim X(0)}\mathbb{E}_{v\sim X_u(0)} f(v)g(v) \\
&= \mathbb{E}_{v\sim X(0)}\mathbb{E}_{u\sim X_v(0)} f(u)g(u).
\end{aligned}$$

Now, we proceed to check the second item. Recall that $\langle Af, g\rangle = \mathbb{E}_{uv\sim X(1)} f(u)g(v)$. The computation is analogous but edges are replaced by triangles and vertices are replaced by edges.

$$\mathbb{E}_{uv\sim X(1)} f(u)g(v) = \sum_{uvw\in X(1)} f(u)g(v)\frac{1}{3}\Pr[uvw] + f(u)g(w)\frac{1}{3}\Pr[uvw] + f(v)g(w)\frac{1}{3}\Pr[uvw].$$

For simplicity, we only consider the first term on the RHS above. We have

$$\begin{aligned}
\sum_{uvw\in X(1)} f(u)g(w)\Pr[uvw] &= \sum_{v\in X(0)} \sum_{uw\in X_v(1)} f(u)g(w)\Pr[uvw] \\
&= \mathbb{E}_{v\sim X(0)}\mathbb{E}_{uw\sim X_v(1)} f(u)g(w) = \mathbb{E}_{v\sim X(0)}\langle A_v f_v, g_v\rangle.
\end{aligned}$$

□

The second result relates the norms f_v paralled to $\mathbb{1}_v$ with Af.

Claim 1.3 (recursion)**.** *Let $f\colon X(0)\to\mathbb{R}$ and $f_v^{\|} = \langle f_v, \mathbb{1}_v\rangle \cdot \mathbb{1}_v$ for $v\in X(0)$. Then*

$$\mathbb{E}_{v\sim X(0)}\langle f_v^{\|}, f_v^{\|}\rangle = \langle Af, Af\rangle.$$

Proof. First, observe that $\langle f_v, \mathbb{1}_v\rangle = \mathbb{E}_{u\sim X_v(0)} f_v(u) = (Af)(v)$. Thus,

$$\mathbb{E}_{v\sim X(0)}\langle f_v^{\|}, f_v^{\|}\rangle = \mathbb{E}_{v\sim X(0)}(Af)(v)(Af)(v) = \langle Af, Af\rangle.$$

□

Now, we are ready to prove the key result.

Proof of 1.1. Take $f \perp 1$ such that $\langle Af, f\rangle = \gamma$ where γ is the second largest eigenvalue of A.

Decompose f_v as $f_v = f_v^{\perp} + f_v^{\|}$ where $f_v^{\|} = \langle f_v, \mathbb{1}_v \rangle \cdot \mathbb{1}_v$. We have

$$\begin{aligned}
\gamma = \langle Af, f\rangle &= \mathbb{E}_{v\sim X(0)}\langle A_v f_v, f_v\rangle && \text{(localization 1.2)}\\
&\le \mathbb{E}_{v\sim X(0)}\langle f_v^{\|}, f_v^{\|}\rangle + \mathbb{E}_{v\sim X(0)}\lambda\langle f_v^{\perp}, f_v^{\perp}\rangle\\
&= \mathbb{E}_{v\sim X(0)}\langle f_v^{\|}, f_v^{\|}\rangle + \mathbb{E}_{v\sim X(0)}\lambda\langle f_v - f_v^{\|}, f_v - f_v^{\|}\rangle\\
&= (1-\lambda)\mathbb{E}_{v\sim X(0)}\langle f_v^{\|}, f_v^{\|}\rangle + \lambda\mathbb{E}_{v\sim X(0)}\langle f_v, f_v\rangle\\
&= (1-\lambda)\mathbb{E}_{v\sim X(0)}\langle f_v^{\|}, f_v^{\|}\rangle + \lambda\langle f, f\rangle && \text{(localization 1.2)}\\
&= (1-\lambda)\langle Af, Af\rangle + \lambda && \text{(recursion 1.3)}\\
&= (1-\lambda)\gamma^2 + \lambda.
\end{aligned}$$

Note that $\gamma = 1$ and $\gamma = \lambda/(1-\lambda)$ are the solutions of the quadratic equation $\gamma = (1-\lambda)\gamma^2 + \lambda$. Under our assumption that the skeleton is connected, we have $\gamma < 1$ and thus $\gamma \le \lambda/(1-\lambda)$. □

The general condition follows from a simple induction using 1.1.

Theorem 1.4 (Strong)**.** *Suppose X is a simplicial complex of dimension $d \ge 2$ endowed with a measure Π_d on the top facets. If*

(i) the second largest eigenvalue of every link X_S for $S \in X(d-2)$ is bounded by λ, and

(ii) every link X_S for $S \in X(\le d-3)$ is connected,

then the skeleton has second largest eigenvalue bounded by $\lambda/(1-(d-1)\lambda)$.

Proof. Let $\lambda_i = \lambda/(1-i\lambda)$. Using induction and 1.1, for $i = 0, \ldots, d-2$, we can upper bound the second largest eigenvalue λ_i of links of co-dimension $d-i+1$ by

$$\frac{\lambda_i}{1-\lambda_i} = \frac{\lambda}{1-i\lambda}\frac{1}{1-\lambda/(1-i\lambda)} = \lambda_{i+1}.$$

□

References

[1] Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders part i: Descent of spectral gaps. *Discrete & Computational Geometry*, 59(2):293–330, Mar 2018.