High-Dimensional Expanders Research Group

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Roughly speaking high-dimensional expanders (HDXs) are hypergraph expanders having similar properties to complete simplicial complexes. This is analogous to the 1-dimensional case in which expander graphs have similar properties to complete graphs. Over the past 40 years expander graphs flourished as an important field at the intersection of theoretical computer science (TCS) and mathematics [HLW06]. Building on top of this rich expander graph theory, the young field of HDXs has already been instrumental in recent breakthroughs (e.g. counting bases of matroids [ALGV19], derandomizing direct product tests [DK17], Gromov's topological overlap property [KKL16], etc) and there is a strong hope that HDXs may turn out to be fundamental objects as their one dimensional analogues. Currently, there is no shortage of open problems about HDXs ranging from their very construction (we only know a few of them) to applications in coding theory, optimization and hardness of approximation.

An ambitious line of work in the TCS community is to use HDXs to achieve three goals: linear size Probabilistically Checkable Proofs (PCPs), completely explicit PCP constructions and random PCP constructions.

In this line of work some natural questions arise (according to the author's biases). Towards the first goal it might be instructive to address the following problems.

1.1 When are CSPs on HDXs hard for our best algorithms (e.g. Sum-of-Squares)? The answer might suggest good parameters for PCPs on HDXs.

1.2 What codes can be obtained from HDXs? In the classical PCP theory, codes and PCPs are intimately connected.

1.3 How do these codes can be used for hardness?

The second goal might involve the following problems.

2.1 What combinatorial constructions of HDXs can we obtain?

2.2 Is there a generalized zig-zag [RVW00] product for HDXs?

2.3 How does representation theory help?

The third goal could be a major breakthrough in average case hardness. It might benefit from answers to the following questions.

3.1 What are good random models for HDXs? This problem seems surprisingly more involved than the graph case.

3.2 Are random CSPs coming from these models hard for our best algorithms (e.g. Sum-of-Squares)?

Another question is: what HDX partitioning can be obtained assuming HDX special property? The one dimensional theory is rich with such results [KLL+13].

1 HDX and Related Papers

In the following we briefly describe some papers about or related to HDXs. Lubotzky has a HDX survey in ICM 2018 [Lub18]. Please note that this list of papers is intended to serve only as starting point for our reading group, but we do not need and should not be restricted to it.

1.1 HDXs Applications to TCS

The HDX field started in pure mathematics around 2005 and only very recently has seen applications in TCS. This freshness provides a variety of research opportunities.

1.1.1 Agreement Testing

- In [KM20], Kaufman and Mass develop a technique dubbed the variance method to derive agreement expasion results.
- In [DD19], Dikstein and Dinur generalized the agreement test on HDXs [DK17] to more general families of "layered" set systems.
- In [DK17], Dinur and Kaufman showed that HDXs are "agreement expanders". As a consequence HDXs can be used to derandomize product test which is an important test in the PCP literature. (Presented by Fernando.)
- In [DFH19], Dinur et al. study higher-order agreement testing generalizing direct product testing. Their generalized agreement testing is applied to p-biased Boolean function applications.

1.1.2 Coding Theory

- In [DDHRZ20], Dikstein et al. propose an approach towards good LTC using HDXs. They also use HDX as a unifying language for LTCs.
- In [JQST20], Jeronimo et al. provide unique decoding algorithm for ϵ-balanced codes near the Gilbert–Varshamov bound. (Presented by Fernando.)
- In [EKZ20], Evra et al. use HDXs to obtain LDPC quantum codes with distance larger than $\sqrt{n}$.
- In [AJQ+20], Alev et al. show how to list decode direct sum codes on HDXs and expander walks. (Presented by Fernando.)
- In [DHK+19], Dinur et al. show how to list decode product codes using double samplers. Currently, the only known constructions of double samplers are based on HDXs. (Presented by Dylan.)

- In [DHKR19], Dinur et al. use agreement testing to make local testing "robust" of certain codes.

1.1.3 Constraint Satisfaction Problems

- In [AJT19], Alev et al. show how to approximate k-CSPs on HDXs using the Sum-of-Squares hierarchy. This result can be viewed as a generalization to k-CSPs (for $k \geq 2$) of the work of Barak, Raghavendra and Steurer [BRS11] for approximating 2-CSPs on low threshold rank graphs.

1.1.4 Counting and Sampling

- In [ALG20], Anari et al. use distributions yielding HDXs to sample independent sets from the hardcore model.

- In [AL20], Alev and Lau give sharp spectral bounds for the up and down walks improving the analysis of [KO18b]. As an application, they show how to sample independent sets.

- In [ALGV19], Anari et al. presented an efficient scheme to count bases of a matroid solving a longstanding open problem of Mihail and Vazirani. (Presented by Akash.)

1.1.5 Sparsification

- In [BST19], Bansal et al. explore new notion of graph and hypergraph sparsifier.

1.2 HDXs Applications to Mathematics

- In [KM18], Kaufman and Mass obtained good distance lattices using high-dimensional expanders.

1.3 HDXs Properties

- In [KO19], Kaufman and Oppenheim study a higher-order notion of edge expansion.

- In [DDFH18], Dinur et al. introduce a machinery of expanding posets clarifying the spectral theory of an important class of HDXs. Their machinery shows that natural random walks on these HDXs behave similarly to their counterpart in the complete complex. (Presented by Shashank.)

- In [KO18b], Kaufman and Oppenheim obtain similar spectral results to those obtained in [DDFH18]. However, the language adopted in these two papers is different.

- In [PRT16], Parzanchevski et al. develop isoperimetric inequalities for HDXs (e.g. they have a notion of high-dimensional Cheeger's inequality).

- In [KM17], Kaufman and Mass (KM) present some notion of high-dimensional expansion. Spectrally, their result is weaker than [DDFH18] and [KO18b]. However, KM result has a stronger combinatorial flavor.

- In [KKL14, KKL16], Kaufman et al. address a conjecture of Gromov about a property called topological overlap which generalizes geometric overlap.
- In [EK16], Evra and Kaufman provide constructions of topological expanders of every dimension generalizing [KKL14].
- In [KL14], Kaufman and Lubotzky show that certain notions of high-dimensional expansion are equivalent to testability of some properties.

1.4 HDXs Constructions

- Harsha and Saptharishi [HS19] provide a self-contained exposition of Kaufman–Oppenheim HDX construction, which avoids representation theory.
- Karni and Kaufman [KK20] use a connection with zig-zag product to improve the expansion of some 3-uniform hypegraph constructions (e.g., Conlon and CLP).
- Liu et al. [LMY19] gave a combinatorial construction of HDXs using expander graphs as a starting point.
- Conlon et al. [CTZ18] gave a combinatorial construction of HDXs with an **almost** linear number of hyperedges. Currently, it might be the simplest non-trivial construction of HDXs.
- Chapman et al. study and construct expander graphs whose neighborhood of every vertex is also expanding [CLP19].
- In [KO18a], Kaufman and Oppenheim give a linear size construction of one-sided link HDXs. The construction is algebraic and uses representation theory. (Presented by Tushant.)
- In [LSV05b, LSV05a], Lubotzky et al. construct Ramanujan complexes generalizing Ramanujan graphs. Their construction is algebraic.

1.5 HDXs Random Models

- In [LLR18], Lubotzky et al. present the first bounded degree random model of HDXs.
- In [LM06], Linial and Meshulam present a dense random model of HDXs similar to the $\mathcal{G}_{n,p}$ random graph model.

1.6 Complete Complex

Roughly, HDXs provide sparse approximations to complete complexes. In some cases, understanding properties of complete complexes may be important towards understanding HDXs. Contrary to complete graphs, properties of complete complexes can be far from trivial.

- In [Fil16], Filmus develop a theory of Boolean functions over a slice of the Boolean cube. Part of his result can be seen as a complete complex analogue of [DDFH18].
- In [DS14], Dinur and Steurer show direct product tests for complete complexes. Their result is used as black box in [DK17]. (Presented by Mrinal.)

- In [DDG+15], Dinur et al. show a direct sum test for complete complexes. Similarly, their result is used as black box in [DK17].

1.7 Classical Results

- In [CY20], Cohen and Yankovitz use expanders to perform query-efficient distance amplification of LDCs.
- Ta-Shma gives an explicit almost optimal (i.e., close to the GV bound) construction of epsilon-balanced codes in [TS17]. (Presented by Shashank.)
- In [MOP19], Sidhanth, O'Donnell and Paredes construct explicit near Ramanujan graphs of every degree.
- By "derandomizing" the long code, Barak et al. proved that a graph family is a small set expander but has reasonably large threshold rank [BGH+12]. These properties were then used to show hardness results.
- In [GLSS18], Garg et al. show a matrix Chernoff bound for expander graphs.
- Finding forbidden minors: one-sided [KSS18] and two-sided testers [KSS19]. (Presented by Akash.)
- Expander graph survery [HLW06] of Hoory, Linial and Wigderson.
- In [IKW09], Impagliazzo et al. give a PCP construction using direct product. Further derandomizing their construction using HDXs may yield linear size PCPs.
- In [RVW00], Reingold et al. give a combinatorial construction of HDXs using a graph operation called zig-zag product which has connections to semi-direct product on groups [Lub01]. It is an open problem to find combinatorial constructions of HDXs.
- In [Din06], Dinur gives an influential combinatorial PCP using expander graphs. Her ideas might be useful in obtaining a linear size PCP using HDXs.
- In [KLL+13], Kowk et al. provide graph partitioning based on higher eigenvalues generalizing Cheeger's inequality.
- Some natural walks in the complete complex are related to the Johnson scheme [GM15].
- Spherical buildings [Rom09] arise as substructures of Ramanujan complexes.

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