HDX Cluster - Open Problem Session 1 *

October 13, 2019

Abstract

This document collects the conjectures, questions and an exercise discussed in the first open problem session of the high-dimensional expander cluster. Each section of the document starts with the statement of the problem followed by definitions (if necessary) and a brief context. For further and more precise details the reader is referred to the associated bibliography.

1 Problem (suggested by Nati Linial)

Conjecture 1.1 (Erdős Conjecture [8]). *There exists STS with arbitrarily large girth.*

Definition 1.1 (STS). *A Steiner Triple System (STS) is a collection of triples that contain every pair exactly once.*

Example 1.1 (Fano Plane). *The Fano plane is an example of STS.*

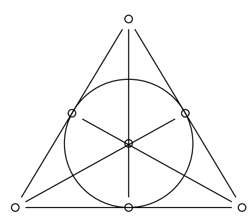

Figure 1: Fano Plane.

Definition 1.2 (Girth). *The girth of a STS is the smallest integer $k \geq 4$ such that there exists some k vertices spanning $\geq k-2$ faces (triples).*

Theorem 1.2 (Kirkman [12]). *There exists a Steiner triple system of order n if and only if $n \equiv 1, 3 \pmod 6$.*

Theorem 1.3 (Glock-Lo–Lühn-Osthus [9]). *There exists partial STS of $(1/6 - \epsilon)n^2$ triples with arbitrarily large girth.*

*This notes were collected and edited with Siddharth Bhandari.

2 Problem (suggested by Nati Linial)

Conjecture 2.1 (Signing Conjecture [1] (Bilu and Linial))**.** *Every d-regular graph has a signing with spectral radius $\leq 2\sqrt{d-1}$.*

Definition 2.1 (Signing)**.** *Let A_G be the adjacency matrix of a d-regular graph G. A signing $\tilde{A}_G$ of A_G is a symmetric matrix (with entries in $\{-1,0,1\}$) satisfying $(A_G)_{i,j} = |(\tilde{A}_G)_{i,j}|$.*

Definition 2.2 (2-lift [1])**.** *Let $G = (V, E)$ be a d-regular graph and let $\tilde{A}_G$ be a signing of G. A 2-lift $H = (V', E')$ of G is a graph on vertex set $V' = V \times [2]$ such that for each $\{u, v\} \in E$ we connect:*

- *$(u,1)$ to $(v,1)$ and $(u,2)$ to $(v,2)$ if $\tilde{A}_G(u,v) = +1$*
- *$(u,1)$ to $(v,2)$ and $(u,2)$ to $(v,1)$ if $\tilde{A}_G(u,v) = -1$.*

Definition 2.3 (Spectral Radius)**.** *Let A be a matrix. The spectral radius of A is the largest singular value of A.*

Remark 2.1. *$Spec(H) = Spec(G) \cup Spec(\tilde{A}_G)$.*

Theorem 2.2 (Bilu and Linial [1])**.** *Every d-regular graph has a signing with spectral radius $O\left(\sqrt{d}\log^{3/2}(d)\right)$.*

Theorem 2.3 (Marcus-Spielman-Srivastava [15])**.** *Signing conjecture is true if the graph is bipartite.*

3 Problem (suggested by Elena Grigorescu)

Question 3.1. *Is there a high-dimensional analogue of δ-local expanders (that might be useful for coding applications)?*

Definition 3.1 (δ-expander)**.** *(A, B) is a δ-expander if for every $X \subset A$ and $Y \subset B$ of relative size δ there exists an edge between X and Y.*

For a directed acyclic graph (DAG) $G = (V, E)$, we assume w.l.o.g. that $V = [n]$ and the standard order of the integers is a valid topological ordering of V.

Definition 3.2 (δ-local expander)**.** *A DAG $G = (V, E)$ is δ-local expander if for every vertex $v \in V$, and every radius r $(A = [v-r+1, v], B = [v+1, v+r])$ is a δ-expander.*

Theorem 3.2 (Erdős-Graham-Szemerédi)**.** *For any $\delta > 0$, there exists a δ-local expander on n vertices with in/out degree $O(\log(n))$.*

In the works of Blocki et al. [3, 2], δ-local expanders were used to obtain a "relaxed" notion of locally correctable codes naturally named relaxed locally correctable codes (RLLC). In the following, we present the gist of how δ-local expanders arise in such constructions.

First, we need to introduce a special kind of hash function. Suppose $H \colon \{0,1\}^* \to \{0,1\}^*$ is a collision resistant hash function (CRHF), i.e., for every probilistic polynomial time (PPT) adversary $\mathcal{A}$ we have $\Pr[\mathcal{A}$ witnesses $H(x) = H(x')]$ is negligible.

Let G be a δ-local expanding DAG on n vertices and $m = m_1 \ldots m_n \in \Sigma^n$ be a message. Using m, we assign bit strings to the vertices of G obtaining a function $\ell \colon V \to \{0,1\}^*$ as follows. For $i = 1 \ldots n$, set $\ell(v_i) = H(m_i \circ \ell(p_1) \circ \cdots \circ \ell(p_r))$ where $\ell(p_1) \circ \cdots \circ \ell(p_r)$ is the concatenation of the strings assigned to the parents $p_1, \ldots, p_r$ of v in G. To encode the message we rely on a

[1]A 2-lift is particular case of a labeled extended graph with binary alphabet as studied in Computer Science in the context of Unique Games.

good code that is efficiently decodable (e.g. Justesen code). Let Enc be the encoding function of this good code. Then, m is encoded as

$$\mathrm{Enc}(m_1)\cdots\mathrm{Enc}(m_n)\ \underbrace{\mathrm{Enc}(\ell(v_1))\cdots\mathrm{Enc}(\ell(v_n))}_{\text{Encoding using }\delta\text{-local expander}}\ \underbrace{\mathrm{Enc}(\ell(v_n))\cdots\mathrm{Enc}(\ell(v_n))}_{\text{Redundancy for }\ell(v_n)}$$

The analysis of the construction relies on the following property.

Definition 3.3 (α-good)**.** *Let $S \subset V$. We say that $v \in V$ is α-good with respect to S if for any radius r*

$$|S \cap [v-r+1, v]| \le \alpha \cdot r \text{ and } |S \cap [v, v+r-1]| \le \alpha \cdot r.$$

4 Problem (suggested by Yuval Filmus)

Question 4.1. *Grassmann-HDX? Known constructions of HDXs derived from [14, 13] provide sparse (linear number of hyperedges in the number of vertices) approximations to the Johnson scheme. How sparse can we make the Grassmann scheme while retaining some of its original properties?*

5 Problem (suggested by Yuval Filmus)

Question 5.1. *What other properties of HDXs can be useful for applications (e.g., locally CAT(0), modified log-sobolev inequality, etc. . .)? Can we go beyond link spectral expansion (c.f. [6, 11, 4])?*

6 Exercise (suggested by Yuval Filmus)

Exercise 6.1 ($(\star, \star, \star, \ldots ?)$)**.** *Let $X(k)$ be the collection of k-faces of a HDX with $k = O(1)$. Does $X(k)$ support a (sound) agreement test with local functions of the form $\{f_S \colon \binom{S}{\le d} \to \Sigma\}_{S \in X(k)}$?*

Definition 6.1 (Agreement Test (Informal))**.** *Pick a set T of size $k/2$ at random and two extensions S_1, S_2 of size k at random. Accept iff $f_{S_1}|_{S_1\cap S_2} = f_{S_2}|_{S_1\cap S_2}$.*

Theorem 6.2 (Dinur and Steurer [7] (DS))**.** *If X is the complete complex and*

$$\Pr\left[f_{S_1}|_{S_1\cap S_2} = f_{S_2}|_{S_1\cap S_2}\right] \ge 1-\epsilon,$$

then there exists $g \colon [n] \to \Sigma$ such that $\Pr\left[f_S = g|_S\right] \ge 1 - O(\epsilon)$.

Theorem 6.3 (Dinur and Kaufman [6])**.** *If X is a HDX, then DS Theorem also holds.*

Theorem 6.4 (Dinur-Filmus-Harsha [5])**.** *If X is the complete complex, then DS Theorem holds for local functions of the form $\{f_S \colon \binom{S}{\le d} \to \Sigma\}_{S\in X(k)}$.*

The Exercise 6.1 can be restated more succinctly as follows.

Exercise 6.5 (Restatement of Exercise 6.1)**.** *Can Theorem 6.4 be generalized to HDX?*

7 Problem (suggested by Sai)

Question 7.1. *Characterize (through HDX?)*

- *Cooperative repairs* [2];
- *Regenerating codes* [3].

Definition 7.1. *We say that a linear code $\mathcal{C}$ has parameters $[n, k, d]_q$ provided it is a code over $\mathbb{F}_q$ with block length n, dimension k and distance d.*

Definition 7.2 (MDS code)**.** *A code $[n, k, d]_q$ is called Maximum Distance Separable (MDS) if $d = n - k + 1$.*

Definition 7.3 (Vectorized MDS Code)**.** *Let $\mathcal{C}$ be a $[n, k, d]_q$ MDS code. We can obtain a vectorized MDS code by replacing each symbol in $\mathbb{F}_q$ by an ℓ-length vector over $\mathbb{F}_t$.*

Consider the case in which a codeword $c_1 \dots c_n \in \left(\mathbb{F}_t^\ell\right)^n$ in a vectorized MDS from a $[n, k, d]_q$ MDS code is distributed among n machines so that the i-th machine stores c_i. Suppose $h \leq n-k$ machines fail (h is typically a constant in applications) causing the erasure of $c_1, \dots, c_h$. To recover $c_1, \dots, c_h$ two rounds of communication are established. In the first round, machines $h+1, \dots, n$ send symbols to the machines $1, \dots, h$. In the second round, machines $1, \dots, h$ send symbols to all $1, \dots, n$ machines. The topology of the communication network in each round is depicted in Fig. 2. Note that it follows the structure of a complete bipartite graph (which can be somewhat dense).

Question 7.2. *Can HDXs be used to derandomize/sparsify the topology of the communication network?*

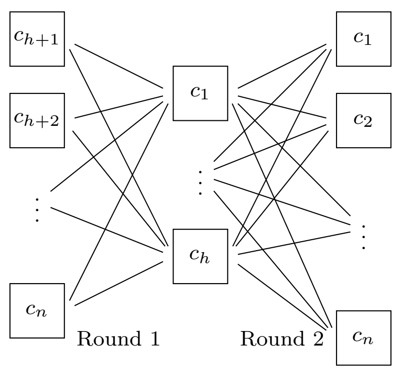

Figure 2: Dense model of communication using complete bipartite graphs.

8 Problem (suggested by David Zuckerman)

Question 8.1. *Is it possible to improve the expander mixing lemma for small sets? Can we obtain explicit k-expanding d-regular graphs on n vertices with $d = O(n/k)$.*

We recall the expander mixing lemma (EML) next.

[2]See reference [17].
[3]See reference [10].

Lemma 8.2 (Expander Mixing Lemma)**.** *Let $G = (V, E)$ be a d-regular graph on n vertices with $\lambda = \max(\lambda_2, |\lambda_n|)$. For every $S, T \subset V$,*

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda\sqrt{|S||T|},$$

where $e(S,T) \coloneqq |\{e = \{s,t\} \in E \mid s \in S, t \in T\}|$.

Improving EML can be useful in the study of k-expanding graphs which is defined next [4].

Definition 8.1 (k-expanding graph)**.** *We say that a graph $G = (V, E)$ is k-expanding provided for every $S, T \subset V$ with $|S|, |T| \geq k$ there exists an edge $\{s, t\} \in E$ with $s \in S$ and $t \in T$.*

Fact 8.3. *Random d-regular graphs with $d = O\left(n \log(n/k)/k\right)$ are k-expanding.*

Theorem 8.4 (Wigderson and Zuckerman [16])**.** *There exists explicit k-expanding d-regular graphs on n vertices with $d = n^{1+o(1)}/k$.*

Remark 8.5. *Harsha suggested that reverse hypercontractivity may be related to Question 8.1.*

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