Construction of one-sided HDX

1 Introduction

The goal of the exposition is to provide an almost completely self-contained proof of the construction of one-sided high dimensional expander as in [KO18]. The paper uses many results from various sources as black-boxes and the aim is to reduce most of those dependencies. The paper also discusses the symmetries of the constructed complex but we will not discuss that as it has no bearing on the construction but is rather a useful property of it. The required representation theory is not much and will be introduced, however, a couple of results will be assumed.

2 Basic Definitions

Before we get into what a high dimensional expander (HDX) is, we need a few preliminary definitions. The natural generalisation of graphs is hypergraphs and a simplicial complex is a hypergraph with more structure which basically let's us "traverse" it more easily.

1. **(Abstract) Simplicial Complex** - Given a base set S, a simplicial complex $X \subset 2^S$ is a collection of subsets which are downward closed, i.e. $\forall A \in X, B \subset A \implies B \in X$. The elements of X are called *simplices* and we denote by $X(i)$ the set of simplices of size $i+1$. $X(0)$ are called the vertices and $X(1)$, the edges. We also have a $X(-1) = \{\phi\}$. The largest n such that $X(n)$ is non-empty is called its *dimension*.

2. **Pure** - An $n-$dimensional simplicial complex X is pure if $\forall A \in X,\ \exists B \in X(n), A \subset B$

3. **Link -** For a simplex τ, it's link $X_\tau = \{\sigma \mid \sigma \cap \tau = \phi,\ \sigma \cup \tau \in X\}$. This basically amounts to looking at the simplex τ, and looking at the complex above it, i.e. by removing everything below it (including it). X_τ is a simplicial complex on $S \backslash \tau$ because if $\sigma_1, \sigma_2 \in S\backslash\tau,\ \sigma_1 \subset \sigma_2,\ \sigma_2 \in X_\tau$, then, by definition, $\sigma_1 \cap \tau = \phi$. Clearly, $\sigma_1 \cup \tau \subset \sigma_2 \cup \tau$ but $\sigma_2 \in X_\tau \implies \sigma_2 \cup \tau \in X \implies \sigma_1 \cup \tau \in X$ (as X is a simplicial complex) and now by the definition of $X_\tau,\ \sigma_1 \in X_\tau$. If the dimension of the complex is n and $\tau \in X(i)$, and $dim(X_\tau) = n - i - 1$.

4. **Connected -** The *1-skeleton* of X is the graph $G = (X(0), X(1))$. X is connected if this 1-skeleton is. The 1-skeleton is important as we will define the spectral properties of the complex in terms of these graphs.

5. **Strongly Gallery Connected**[1] **-** X is *strongly gallery connected* if X is connected and all its links are connected.

[1] This is not the actual definition but we as well might take it as it is equivalent and the original definition is of no use here

3 HDX - Overview

3.1 Concrete Definitions

Definition 1. *For each $k \leqslant n-2$, $\tau \in X(k)$, define μ_τ to be the second largest eigenvalue of the weighted 1-skeleton:*

$$\{u,v\} \in X_\tau(1),\ w(u,v) = (n-k-2)!|\{\sigma \in X_\tau(n-1-k) \mid \{u,v\} \subset \sigma\}|$$

Note that for the 1-D link i.e. $k = n-2$, this is precisely the unweighted graph.

There are other definitions of HDX which include a distribution D which defines the weights on the top layer(i.e $X(n)$) and induces it downwards uniformly. These set of weights correspond to the uniform distribution on the top layer.

Definition 2. *For $0 \leqslant \lambda < 1$, a pure n-dimensional finite simplicial complex X is a one-sided λ-local-spectral expander if for every $-1 \leqslant k \leqslant n-2$, and $\forall \tau \in X(k)$, $\mu_\tau \leqslant \lambda$.*

This means that the 1-skeleton of every link is an expander. This requires them to be connected in the first place and that is why the strongly gallery connected property is needed. Moreover, *purity* is implicitly used in defining the weights as the sets in $X(i)$ get weights induced from $X(i-1)$ but that doesn't work if for some set there is no set containing it. The purity property prevents such a thing from happening.

3.2 The problem to solve

The goal is, given $0 < \lambda < 1$ and $n > 1$, to construct a family of pure n-dimensional finite simplicial complexes $\{X^{(s)} \mid s \in A\}$ where $A \subset \mathbb{N}$ is an infinite set such that the following holds -

- **Expansion** - For every $s \in A, X^{(s)}$ is a one-sided λ-local-spectral expander.
- **Bounded Degree** - Every vertex is contained in a bounded number of n-dimensional simplices, i.e., $\exists Q, \forall s \in A, v \in X^{(s)}(0),\ |\{\sigma \in X^{(s)}(n) \mid v \in \sigma\}| < Q$
- **Unbounded vertices** - $\lim_{s\to\infty} |X^{(s)}(0)| = \infty$

3.3 Overview of the plan

This is how the entire flow of the rest of the paper is going to be.

1. Give a general mechanism to construct a complex X using a group G and set of it's subgroups K_i.
   (a) Define a set of axioms that we want (G, K_i) to obey.
   (b) Show that these axioms imply that X is pure and strongly gallery connected.
2. Define the group of elementary matrices and its subgroups we need, and prove that they satisfy the above mentioned axioms.

3. To prove that it is HDX, we start with the expansion property which is the crux. Here we use the result from [Opp18] which states that if the 1-D links expand then all links expand.

4. To show expansion of 1-D links, we define another property called *orthogonality*

   (a) Prove that orthogonality implies expansion.

   (b) Finally, we prove that the 1-D links are orthogonal.

5. We would also need to show the other 2 requirements i.e. bounded degree and unbounded vertices and that would be easy to do.

4 Subgroup Geometry Systems

4.1 Coset Complex

Given a group G and a collection of subgroups $(K_i)_{i\in I}$, we want to construct a simplicial complex X. Define the cosets of K_i as the set formal elements $\{gK_i \mid g \in G\}$ but in which we identify equal sets i.e. $gK_i = hK_i$ if they are equal as sets, this is equivalent to saying that $h^{-1}g \in K_i$. Define a complex X over the base set $\{gK_i \mid g \in G,\ i \in I\}$ in which all the cosets are the vertices and $(g_0K_0, \cdots, g_lK_l) \in X(l) \iff \forall i, j \leqslant l,\ g_iK_i \cap g_jK_j \neq \phi$. Note that this implies that no 2 subgroups can be same because if $K_i = K_j$ then $g_iK_i \cap g_jK_j \neq \phi \implies g_iK_i = g_jK_j$. This X so formed is denoted as $X(G, (K_i)_{i\in I})$. We now need the set of axioms that will ensure that this complex is "nice".

4.2 SGS axioms

Let $\tau \subset I$ and define $K_\tau = \cap_{i\in\tau} K_i$. Also define $K_\phi = G$. $(G, (K_i)_{i\in I})$ is called a subgroup geometry system (SGS) if it obeys the following 3 axioms

1. $\forall \tau, \sigma \subset I,\ K_{\tau\cap\sigma} = \langle K_\tau, K_\sigma \rangle$ where $\langle\rangle$ denotes the subgroup generated by the union

2. $\forall \tau \subset I,\ i \notin \tau,\ K_\tau K_i = \cap_{j\in\tau} K_jK_i$

3. $\forall i \in I,\ K_I \neq K_{I\setminus\{i\}}$

The axioms 1 and 3 weed out boundary cases like $K_i \neq G, \{e\}$ because say $K_i = G$, then $K_I = K_{I\setminus\{i\}} \cap G = K_{I\setminus\{i\}}$ which violates axiom 3. Similarly, if $K_i = K_j$ or if $K_j = \{e\}$, then by axiom 1 $G = K_\phi = K_{\{i\}\cap\{j\}} =< K_i, K_j >= K_i$ but that we have seen is not allowed. The main axiom is then the second one. One direction is trivial i.e. $K_\tau K_i \subset \cap_{j\in\tau} K_jK_i$ because if $xy \in K_\tau K_i,\ x \in K_\tau, y \in K_i$ then $x \in K_j \forall j \in \tau$ and thus $xy \in K_jK_i$. But the other is not and it essentially is like a local-global (or a glueing) property which says that if we have elements in the local pairwise product we have one in the overall intersection which is in the product. The proofs below will highlight the use of this property.

Note - As an abuse of notation, when $X = X(G, (K_i)_{i\in I})$ and $(G, (K_i)_{i\in I})$ is an SGS, we will simply write that the complex X is an SGS.

4.3 Properties that SGS implies

As mentioned earlier, a necessary property that we need to build an expander is purity and strongly gallery connectedness. We shall now prove that if X is an SGS, these properties are ensured. We'll prove a couple more which we'll need later.

4.3.1 Uniformity

This is the most important property and we shall see implies most of the other ones. It crucially uses Axiom 2.

Lemma 1. $\forall l \leqslant n,\ \forall \sigma = (g_1 K_{i_1}, \cdots, g_l K_{i_l}) \in X(l), \exists g,\ \sigma = (gK_{i_1}, \cdots, gK_{i_l})$

Proof. We prove this by induction on l.
Base Case - $l = 1$, Trivial
Inductive Step - Assume true for l.
$(g_1 K_{i_1}, \cdots, g_l K_{i_l}, g_{l+1} K_{i_{l+1}}) = (gK_{i_1}, \cdots, gK_{i_l}, g_{l+1} K_{i_{l+1}})$ Now, as $\sigma \in X(l)$, by definition each pair must intersect. Thus, $\forall i \in [l],\ \exists k_i \in K_i, m_i \in K_{l+1}\ gk_i = g_{l+1} m_i$ and thus, $g^{-1} g_{l+1} \in K_i K_{l+1}$. By axiom 2, $\exists h \in K_{1,\cdots,l}\ , m \in K_{l+1},\ hm = g^{-1} g_{l+1} \implies gh = g_{l+1} m^{-1}$. Define $g^* = gh$. Clearly $\forall\ i \leqslant l,\ g^* K_i = ghK_i = gK_i$ as $h \in K_i$ and $g^* K_{l+1} = g_{l+1} m^{-1} K_{l+1} = g_{l+1} K_{l+1}$ as $m \in K_{l+1}$. □

4.3.2 Transitivity of top layer

Trivial from above, to map $\sigma \to \tau$, uniformize both σ, τ. Then multiply by gh^{-1} where g, h are the uniform elements of σ, τ respectively.

4.3.3 Purity

Let $\sigma \in X(k)$. By uniformity $\sigma = g(K_1, \cdots, K_l)$. Now, $\forall g,\ g(K_0, \cdots, K_n) \in X(n)$ as each of the mutual intersection contains g. Thus, we have purity.

4.3.4 Links are Coset Complexes

$X_\tau \cong X(K_\tau, (K(\tau \cup i))_{i \in I \setminus \tau})$ We only need it for $1 - D$ links and we will thus show it when $\tau \in X(n-2)$ but the proof is exactly the same without this assumption.

Proof. The 1-D links are basically graphs and we need to construct a graph isomorphism. That is a bijective mapping between vertices such that edge relations are preserved. WLOG, assume $\tau = \{2, \cdots, n\}$. $X(K_\tau, (K(\tau \cup i))_{i \in I \setminus \tau}) = (K_{2\cdots n}, (K_{02\cdots n}, K_{12\cdots n})))$. We have an easy inclusion in one direction, $g' K_{02\cdots n} \to (K_2, \cdots K_n, g' K_0) = (g' K_2, \cdots g' K_n, g' K_0)$ This is valid as, $g' = g' \cdot e \in g' K_0$ but $g' \in K_i, \forall i \in \{2, \cdots n\}$. We need that this is surjective. Let $\sigma = (K_2, \cdots K_n, gK_0)$. If $g \in K_{2\cdots n}$ we would be done as it would be the image of $gK_{02\cdots n}$. If not, since its is a simplex we have pairwise intersections. Thus, $\forall i \in \{2 \cdots n\},\ \exists k_i \in K_i, l_i \in K_0,\ k_i = gl_i \implies g = k_i l_i^{-1} \in K_i K_0$. From axiom 2 $g \in K_{2\cdots n} K_0 = g' l$ where $g' \in K_{2\cdots n},\ l \in K_0$. Then, $\sigma = ((K_2, \cdots K_n, g' K_0)$ and we have a preimage $g' K_{02\cdots n}$.

To see that edge relations are preserved, in the coset complex, $(gK_{02\cdots n}, hK_{12\cdots n})$ have an edge iff $h^{-1}g \in K_{12\cdots n}K_{02\cdots n}$.

Let's look at their images. They share an edge only if $(K_2, \cdots K_n, gK_0, hK_1) \in X_\tau \iff h^{-1}g \in K_1K_0$ as we already know that the other intersections are non-empty (each is a simplex). But $g, h \in K_{2\cdots n}$. Thus, $h^{-1}g \in K_1K_0 \iff h^{-1}g \in K_{12\cdots n}K_{02\cdots n}$.

□

4.3.5 Strongly gallery connected

It suffices to show that X is connected because if $X(G,(K_i)_{i\in I})$ is an SGS, we have seen that $X_\tau \cong X(K_\tau,(K_{\tau\cup i})_{i\notin\tau})$ which is also an *SGS*. Thus from the above claim we can deduce that X_τ is connected too. From the lemma we proved in the last section, we will have that X is *strongly gallery connected*

Lemma 2. *If X is an SGS then X is strongly gallery connected*

Proof. Each K_i creates a partition of G which are represented by $\{gK_i \mid g \in G\}$. The 1-skeleton of X is an $|I|-$partite graph and we know that $\forall g, i, j\ (gK_i, gK_j) \in X(1)$ thus if we show that (gK_i, hK_i) is connected then for any (gK_i, hK_j) we have a path from $(gK_i hK_i)$ and an edge between (hK_i, hK_j). Assume for a contradiction that for some i the we have 2 disconnected components $A = \mathbf{g}K_i = \{g_1K_i, \cdots g_lK_i\}$ and $B = \mathbf{h}K_i = \{h_1K_i, \cdots h_mK_i\}$. Now these basically define a partition of G into 2. One easy consequence of the axiom 3, i.e, $K_i \neq K_j$ is that the partitions created are not same. That is for the same sets $\mathbf{g}K_j \neq \mathbf{g}K_i$. Therefore one of the cosets lK_j will intersect both A and B and there are thus at least 1 edge each from lK_j to some coset in A, B. Therefore, the are not disconnected. □

Now we'll define the specific complex which we will eventually prove is the HDX we seek.

5 Elementary Matrices Complex

Given a unital ring R, we consider 2 objects over it using a generating set $\{t_1, \cdots t_l\}$. One, is another ring i.e. a finitely generated $R-$algebra $\mathcal{R}$ and the other is a free $R-$module T generated with $\{1, t_1, \cdots, t_l\}$ as a basis.

We can think of $\mathcal{R}$ as T equipped with a multiplication which is associative and distributive over addition, this gives us a tensor product algebra kind of an interpretation. Another way is to view it as a ring of polynomials in $t_i's$ modulo some relation. If there are no relations, then it is the free *nonunital* algebra which is also called the ring of words (with no empty string). If we add a unit 1 to the list of generators and impose the relations that $1t_i = t_i1 = t_i\ i \in l$, we get the free *unital* algebra also called the non-commutative polynomial ring $\mathbb{R} = \ R < t_1, \cdots t_l > \cong \sum_{i\in\mathbb{N}} T^{\otimes i}$. Note that, $T^{\otimes i}$ is the $R-$module of all (non-commuting) polynomials of degree $\leqslant i$. Further imposing $t_it_j = t_jt_i$ gives us the familiar ring of commutative polynomials. Similarly, we can create algebra of bounded degree polynomials and so on.

The paper concerns only with *unital* $\mathcal{R}$ (though it doesn't explicitly say this) and we can thus think of them as the non-commutative polynomials.

5.1 The Coset Complex

The main group is the (multiplicative) group of elementary matrices,

$$EL_{n+1}(\mathcal{R}) =< e_{ij}(r) \mid 0 \leqslant i, j \leqslant n, r \in \mathcal{R} >$$

where $e_{ij}(r)$ is the matrix with $1's$ on the diagonal, r on the i, j^{th} entry and 0 elsewhere. Clearly, the determinant of this is 1 and thus $EL_{n+1}(\mathcal{R}) \subset SL_{n+1}(\mathcal{R})$. If $\mathcal{R}$ is a Euclidean ring like say, $\mathbb{Z}$, then both are equal.[2] Define the subgroups,

$$K_i =< e_{jj+1}(m) \mid j \in \{0, \cdots n\} \backslash i, m \in T >$$

Note that the subscripts are taken modulo n i.e $e_{nn+1} = e_{n0}$

5.1.1 Steinberg Group

We can instead look at EL_n more abstractly as the Steinberg group generated by $x_{ij}(\mathcal{R})$ which satisfy the Steinberg relations,

$$x_{j_1,j_2}(m_1)x_{j_1j_2}(m_2) = x_{j_1j_2}(m_1 + m_2) \quad (1)$$
$$[x_{j_1,j_2}(m_1), x_{j_2j_3}(m_2)] = x_{j_1j_3}(m_1m_2) \;\; j_1 \neq j_3 \quad (2)$$
$$[x_{j_1,j_2}(m_1), x_{j_3j_4}(m_2)] = I \;\; j_1 \neq j_4, j_2 \neq j_3 \quad (3)$$

Note that in the equations $2, 3$ we have the commutator bracket, $[g, h] := g^{-1}h^{-1}gh$. We have a natural map then $St_n(\mathcal{R}) \to EL_n(\mathcal{R}) : x_{ij}(r) \to e_{ij}(r)$.

6 Elementary Matrices are an SGS

Before we show that they are a SGS, we need a few lemmas characterising the subgroups to make it easier to work with them. We will write T^i to denote $T^{\otimes i}$ which is all polynomials of degree at most i. Denote by $[k, j)$ the interval from k to $j - 1$, i.e

$$[k, j) = \left\{ \begin{array}{cc} \{k, k+1, \cdots, j-1\} & k < j \\ \{k, k+1, \cdots, n, 0, 1, \cdots, j-1\} & k \geqslant j \end{array} \right\}$$

Lemma 3. *K_i is the group composed of matrices $A = (a_{kj})$ such that*

$$a_{kj} \in \left\{ \begin{array}{cc} \{1\} & k = j \\ T^{j-k} & k \neq j, i \notin [k, j) \\ \{0\} & \textit{otherwise} \end{array} \right\}$$

[2] If they are equal for a ring R, R is called general Euclidean. The exact requirements for this to hold are not clear and this has deep connections to K-theory.

Proof. We will prove it only for $i = n$. Notice that K_n is the group of upper triangular matrices. Observe that, $[k, j) \cup [j, k) = \{0, \cdots, n\}$. Thus at most one of a_{kj} can be 0 if the above lemma is true.

Using Steinberg relation (2) we can create by induction, for every $j > k$, $e_{j,k}(M)\, M \in T^{k-j}$ where M are monomials. To add them up, we use the relation (1) and thus get each entry as as polynomials thereby covering all of T^{j-k}. Now to compose all these entries in a single matrix we use relation (3). For example, (blanks denote 0)

$$\begin{bmatrix} 1 & m_1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & m_2 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & m_1 & & \\ & 1 & & \\ & & 1 & m_2 \\ & & & 1 \end{bmatrix}$$

□

Lets now prove the 3 axioms required for it to be an SGS.

Theorem 4. *$(EL_{n+1}(\mathcal{R}), (K_i)_{i\in I})$ is an SGS*

Proof. **Axiom 1** - $K_{\tau\cap\sigma} =< K_\tau, K_\sigma >$. Let S_i generate K_i Then, K_τ is generated by $\cap_{i\in\tau} S_i$. The equivalence thus follows trivially for $\tau \neq \phi$. In that case we need to show that $EL_{n+1}(\mathcal{R}) =< e_{ii+1}(m)\ |i \in [n], m \in T >$. To see this , note that the generators clearly generate a subset $EL_{n+1}(\mathcal{R})$ with $(k, j)^{th}$ entry of degree $\leqslant (j - k) \mod n + 1$. To increase degree we can use relation 2 and thus $[e_{02}(m), e_{21}(m)] = e_{01}(m^2)$

Axiom 2 We want to show that $K_\tau K_l = \cap_{i\in\tau} K_i K_l$

Define the projection morphism $\pi_\phi : EL_{n+1}(\mathcal{R}) \to K_l$ which maps the common generators to itself and the other ones to identity. To make it precise $\pi_\phi(e_{ij}(r)) = \left\{ \begin{matrix} e_{ij}(r) & e_{ij}(r) \in K_l \\ I & \text{otherwise} \end{matrix} \right\}$

This restricts to a map for any σ $\pi_\sigma : K_\sigma \to K_{\sigma\cap l}$.

Lemma 5. *$\pi_\phi : EL_{n+1}(\mathcal{R}) \to K_l$ is a homomorphism*

Proof. To prove it the only thing we need to check is that the relations are preserved.

Relation 1 - $e_{j_1,j_2}(m_1)e_{j_1j_2}(m_2) = e_{j_1j_2}(m_1 + m_2)$
Is trivially preserved as if $e_{ij}(\lambda)$ either remains same or maps to identity. In case one the equation is unaffected in the other case everything becomes identity.

Relation 2 - $[e_{j_1,j_2}(m_1), e_{j_2j_3}(m_2)] = x_{j_1j_3}(m_1m_2)\ \ j_1 \neq j_3$
If both $e_{j_1j_2}, e_{j_2,j_3}$ are in K_l then the equation is unchanged. If even one is not in K_l then using the characterisation of K_l, $l \in [j_i, j_2) \cup [j_2, j_3) = [j_1, j_3)$ and thus $e_{j_1j_3} \notin K_l$. Thus we have identity on both sides.

Relation 3 - $[e_{j_1,j_2}(m_1), e_{j_3j_4}(m_2)] = I\ \ j_1 \neq j_4, j_2 \neq j_3$
Doing the same analysis it's unchanged if both are in K_l if one isn't then the LHS becomes identity as $[g, I] = I$

□

Clearly even the restrictions are homomorphisms and denote their kernel as K_σ^-. Then $\forall g \in K_\sigma,\ gK_l = g\pi(g)^{-1}K_l$ as $\pi(g)^{-1} \in K_l$ but $\pi(g\pi(g)^{-1}) = I$ and thus, $g\pi(g)^{-1} \in K_\sigma^-$. Therefore we can assume that the coset representatives come from K_σ^-. Let us look at the RHS and say $x \in \cap_{i\in\tau} K_i K_l$. We have $x = g_i k_i = g_j k_j\ g_i \in K_i\ i \in \tau\ k_i, k_j \in K_l$. This implies $g_j^{-1} g_i \in K_l$ but $\pi_\phi(g_j^{-1} g_i) = k_j k_i^{-1}$ but by assumption the should map to I and thus $k_i = k_j$ and therefore $g_i = g_j$. Continuing this for all, we have that all are equal and lie in K_τ and thus $x \in K_\tau K_l$. [3]

Axiom 3 - This follows trivially from the description of subgroups by the generators. □

7 Structure of the 1-D links (Heisenberg Groups)

We proved earlier that links are the same as coset complexes generated from K_τ instead of G. In the case of the above complex, the 1-D links are of two kinds depending on which 2 subgroups we choose to 'omit'. First is of the type $H := K_{1,3\cdots n} = \langle K_{123\cdots n}, K_{013\cdots n}\rangle = \langle e_{01}(m), e_{23}(m)\rangle$. The third Steinberg relation says that $e_{01}(m)$ and $e_{23}(m)$ commute and thus the subgroup generated by them commute. The following lemma then gives that the spectrum here in uninteresting, i.e. second largest eigenvalue is 0.

Lemma 6. *If $G = \langle K_0, K_1\rangle$ and K_0, K_1 commute, then the complex $X(G, (K_0, K_1))$ is a complete bipartite graph and therefore, its spectrum is $\{-1, 0, 1\}$.*

Proof. Since G is generated by K_0, K_1 each element $g = \prod g_i$, but since K_0, K_1 commute we can rearrange such that $g = g_0 g_1,\ g_i \in K_i$. From the definition we have that it is a bipartite graph with vertices of type $g_1 K_0$ and $g_0 K_1$ where $g_i \in K_i$. There is an edge if $g_1 K_0 \cap g_0 K_1 \neq \phi$. Clearly, $g_1 g_0 \in g_1 K_0$ but due to commutativity $g_1 g_0 = g_0 g_1 \in g_0 K_1$ and thus the bipartite graph is complete. □

The important case is when $H := K_{2\cdots n} = \langle e_{01}(m), e_{12}(m)\rangle$ which concretely are 3×3 upper triangular matrices of the form,

$$\begin{bmatrix} 1 & T & T^2 \\ & 1 & T \\ & & 1 \end{bmatrix}, \text{ generated by } x = \begin{bmatrix} 1 & m & \\ & 1 & \\ & & 1 \end{bmatrix}, y = \begin{bmatrix} 1 & & \\ & 1 & m \\ & & 1 \end{bmatrix},$$

H is referred to as the 'Heisenberg Group' because of its connections to physics. Throughout the rest of the presentation we will denote $X = K_{12\cdots n} = \langle x\rangle$ and $Y =_{02\cdots n}= \langle y\rangle$. Define the subgroup $Z = \langle [x, y]\rangle$. and it's easy to see that $[X, Z] = [Y, Z] = I$, i.e. Z commutes with every group element. We will now proceed to look at the bounds of the second largest eigenvalue of the graph which is the complex $X(H, (X, Y))$

[3] **Note -** The paper doesn't use this language but rather does it for $l = n$ and maps matrices to their upper triangular half. While I think this approach is the same, it works more generally and the proof is easy to see.

8 Orthogonal $\implies$ Expansion

In this section we will define a notion of orthogonality of subgroups and show that the coset complex of the Heisenberg group expands iff the groups are close to orthogonal.

8.1 Orthogonality for Vector Subspaces

Given a vector space V equipped with an inner product, and a pair of subspaces $U, W \subset V$, define $U' = U \cap (U \cap W)^\perp, W' = W \cap (U \cap W)^\perp$,

$$\theta(U, W) = \sup \left| \frac{\langle x, y \rangle}{\|x\| \|y\|} \right| \quad \forall x \in U', \forall y \in W'$$

U, W are called $\epsilon-$orthogonal if $\theta(U, W) \leqslant \epsilon$. This can also be related to the operator norm of the projection matrices. Say we have orthogonal projections $P_U, P_W, P_{U \cap W}$, then

$$\theta(U, W) \leqslant \epsilon \iff \|P_U P_W - P_{U \cap W}\| \leqslant \epsilon$$

We'll take the projection norm to be the definition of orthogonality. **Should I write a proof for this? **

8.2 A quick dive into Representation Theory

Since we have groups and not vector spaces we need some machinery to make a conversion. This is precisely what representation theory does, it linearizes groups.

Definition 3. *For a group G, a $G-$representation is a tuple (V, ρ) where V is a vector space and $\rho : G \to GL(V)$ is a group homomorphism. More concretely, if V is n-dimensional, the map ρ assigns to each group element g, an $n \times n$ invertible matrix $\rho(g)$ such that $\rho(gh) = \rho(g)\rho(h)$ and $\rho(e) = I_n$. A representation is unitary if every $\rho(g)$ is unitary.*

The space of invariant vectors of a subgroup $H \subset G$ is defined as $V_\rho^H = \{v \mid \rho(g)v = v \ \forall g \in H\}$. This is clearly a vector space and for the case we're interested in, i.e the Heisenberg group, we have 2 subspaces V_ρ^X, V_ρ^Y and since $G = \langle X, Y \rangle$, $V_\rho^G = V_\rho^X \cap V_\rho^Y$. The subgroups X, Y are said to be $\epsilon-$orthogonal if for *every* unitary representation ρ, $\theta(V_\rho^X, V_\rho^Y) \leqslant \epsilon$. Now, we define the projection maps to the respective spaces. These are easy to define if G is finite and are basically the averaging map,

$$P_{V_\rho^X}(v) = \frac{1}{|X|} \sum_{g \in X} \rho(g) v$$

To see that they project is easy $\rho(h) P_{V_\rho^X}(v) = \frac{1}{|X|} \sum_{g \in X} \rho(h)\rho(g)v = \frac{1}{|X|} \sum_{g \in X} \rho(gh)v =$

$\frac{1}{|X|}\sum_{g\in Xh}\rho(g)v = P_{V_\rho^X}(v)$ The only part to verify is that these are orthogonal.

$$\begin{aligned}\langle P_{V_\rho^X}(u), v\rangle &= \frac{1}{|X|}\sum_{g\in X}\langle \rho(g)u, v\rangle \\ &= \frac{1}{|X|}\sum_{g\in X}\langle u, \rho(g^{-1})v\rangle \text{ As } \rho \text{ is unitary} \\ &= \frac{1}{|X|}\sum_{g^{-1}\in X}\langle u, \rho(g^{-1})v\rangle \text{ Just changing order of summation} \\ &= \langle u, \frac{1}{|X|}\sum_{g^{-1}\in X}\rho(g^{-1})v\rangle \\ &= \langle u, P_{V_\rho^X}(v)\rangle\end{aligned}$$

8.3 V_{reg} and why bounding it suffices

It might seem that the ϵ-orthogonality definition is unwieldy as it is quantified over *every* unitary representation. However, akin to the prime numbers and integers, there are small representations which are the building blocks of all representations.

Definition 4 (Irreducible representation)**.** *Given a group G, a representation V, ρ is said to be irreducible if for every proper subspace $0 \neq W \neq V$, $\exists g \in G, w \in W,\ \rho(g)w \notin W$. This essentially means that there is no smaller representation W sitting inside V.*

Theorem 7 (Maschke)**.** *Given a finite group G, there exists a finite set of irreducible representations V_i such that every representation $V \cong \oplus_i V_i^{a_i},\ a_i \in \mathbb{Z}_{\geqslant 0}$*

Thus, to prove ϵ-orthogonality we just need to check a finite list of irreducibles because every other representation breaks down in this way as and by Cauchy-Schwartz, bounding each of the direct sum suffices. But in fact, we can just check one.

Let G be a finite group and define a vector space of all formal sums of group elements i.e. $V_{reg} = \mathbb{C}[G] \cong \mathbb{C}^{|G|}$. Define a basis $\{v_g \mid g \in G\}$. The homomorphism is given by $\rho(g)(v_h) = v_{hg}$ i.e it's a permutation matrix. This is called the (right) regular representation. We use another theorem which says that this is that,

Theorem 8 (Peter-Weyl)**.** $V_{reg} \cong \oplus_i V_i^{\mathsf{dim}(V_i)}$ *where the summation is over all irreducible representations of G*

Putting it all together,

Theorem 9. *X, Y are ϵ-orthogonal iff $\|P_{V_\rho^X}P_{V_\rho^Y} - P_{V_\rho^G}\| \leqslant \epsilon$.*

8.4 Equivalence between orthogonality and eigenvalue

8.4.1 Rephrasing λ

The main goal is to bound the second-largest eigenvalue of a biregular bipartite graph. Say $V = V_0 \cup V_1$ is a birpartite graph with V_i being the 2 components such that each vertex in V_i has degree d_i. We need the seconlargest eigenvalue of its weighted adjacency matrix, i.e., $M_{uv} = \frac{1}{d_i}$ if $u \in V_i, (u,v) \in E$. This matrix can be thought of as a linear map from the (formal)vector space of vertices to itself. Since it is bipartite it can be decomposed as a map between the 2 subspaces corresponding to the 2 components.

Let's now formalize everything. Define the formal vector space $S = \mathbb{C}^|V| = \{\sum_{v \in V} a_v e_v \mid a_v \in \mathbb{C}\}$ [4] We need second largest eigenvalue of the map defined by the matrix $M : S \to S$.

To make it the largest eigenvalue we remove the 2 extreme eigenvectors ($1^{|V|}$ and $(1^|V_0|, -1^{|V_1|})$) which we know have eigenvalues of $(1, -1)$. Define the subspace

$$S' = \{\sum_{v \in V} a_v e_v \mid \sum_{v \in V_0} e_v = \sum_{v \in V_1} e_v = 0\}$$

Note that $S' \cong \mathbb{C}^{|V|-2} \cong \mathbb{C}^{|V_0|-1} \oplus \mathbb{C}^{|V_1|-1} =: S'_0 \oplus S'_1$. The map M restricts to a map $M' : S' \to S'$ and moreover, it switches the components, i.e. let $M'(S'_i) \subset S'_{i+1}$ because $M' = \begin{bmatrix} 0 & M_1 \\ M_0 & 0 \end{bmatrix}$ after reordering the basis.

To be more concrete, for $u \in V_i$, $M_i e_u = \frac{1}{d_{i+1}} \sum_{(u,v) \in E} e_v$ Rescale the usual innerproduct on S'_i as $\langle e_u, e_v \rangle d_i \mathrm{I}[u = v]$

Lemma 10. *Let λ be the second highest eigenvalue of M. Then $\lambda = \|M'\|_* = \|M_i\|_*$ where the norm is the operator norm*[5]

Proof. By construction, the largest eigenvalue of M' is the second largest eigenvalue of M'. Moreover, as M' is symmetric, its operator norm and largest eigenvalue are equal. Thus, $\lambda = \|M'\|_*$

$\langle M_0 e_u, e_v \rangle = d_{i+1} \left(\frac{1}{d_{i+1}} \sum_{(u,w) \in E} \langle e_w, e_v \rangle \right) = \mathrm{I}((u,v) \in E) = \langle e_u, M_1 e_v \rangle$. Therefore, these are adjoint operators and thus their spectral norms are equal.

We have, $\|M'v\| = \|M_0 v_0 + M_1 v_1\| = \sqrt{d_0 \|M_0 v_0\|^2 + d_1 \|M_1 v_1\|^2}$ Taking $v_1 = 0$, we have $\|M'v\|_* \geqslant \|M_0\|_*$ and in the other direction,

$$\sqrt{d_0 \|M_0 v_0\|^2 + d_1 \|M_1 v_1\|^2} \leqslant \|M_0\|_* \sqrt{d_0 \|v_0\|^2 + d_1 \|v_1\|^2} = \|M_0\|_* \|v\|_*$$

□

8.4.2 Reformulating M'

So, now we have our bipartite graph $X(H, (X, Y))$.

The 2 components are $V_X = \{gX \mid g \in G\}$ and $V_Y = \{hY \mid h \in G\}$. Moreover, $hY = h'Y \implies h'h^{-1} \in Y$ and thus for each h there are Y identical elements.Therefore, $|V_X| = |G/X| = \frac{|G|}{|X|}$ and similarly, $|V_Y| = |G/Y| = \frac{|G|}{|Y|}$.

Denote the analogous subspaces as above S'_i by S'_X, S'_Y. We need to compute the norm of matrix $M_X : S'_X \to S'_Y$. Let's restate the vector spaces and what exactly we will be proving.

[4] This can also be identified with the dual and seen as the space of functions from $V \to \mathbb{C}$ and that is what the paper does. However, I find that unnecessarily cumbersome.

[5] $*$ is used to denote that it's scaled and not the usual Euclidean norm

$$\begin{aligned}
V_{reg} &\cong \mathbb{C}^{|G|} = \left\{\sum_{g\in G} a_g e_g\right\},\ V'_{reg} = \left\{\sum_{g\in G} a_g e_g \mid \sum a_g = 0\right\} \\
S_X &\cong \mathbb{C}^{|G/X|} = \left\{\sum_{x\in G/X} a_x e_x\right\},\ S'_X = \left\{\sum_{x\in G/X} a_x e_x \mid \sum a_x = 0\right\} \\
S_Y &\cong \mathbb{C}^{|G/Y|} = \left\{\sum_{y\in G/Y} a_y e_y\right\},\ S'_Y = \left\{\sum_{y\in G/Y} a_y e_y \mid \sum a_y = 0\right\} \\
S' &= S'_X \oplus S'_Y
\end{aligned}$$

To connect this to the ϵ orthogonality we need to relate it to $\|P_{V_\rho^X} P_{V_\rho^Y} - P_{V_\rho^G}\|$. The above is a map on V_{reg} and we need to relate it to the norm of M'. We show the following observation,

Lemma 11. *$S_X \cong V_\rho^X$ and similarly, $S_Y \cong V_\rho^Y$*

Proof. We'll show it just for X as it is symmetric. In one direction, if $g' = g \cdot x_0$ for some $x_0 \in X$, then $P_\rho^X(g') = \sum_{x\in X} g' \cdot x = \sum_{x\in X} g \cdot x_0 \cdot x = \sum_{t\in X} g \cdot t = P_\rho^X(g)$. In the other, if $P_\rho^X(g) = P_\rho^X(h)$, then $\sum_{x\in X} g \cdot x = \sum_{x\in X} h \cdot x$ which means that $\exists g = h \cdot x$. Thus, the image of projection map which is V_ρ^X is in bijection with the space spanned the cosets G/X which by definition is S_X □

Lemma 12. *The following maps are orthogonal projections,*

$$V_{reg} \xrightarrow{I-P_\rho^G} V'_{reg} \xrightarrow{P_\rho^X} S'_X$$

$$V_{reg} \xrightarrow{I-P_\rho^G} V'_{reg} \xrightarrow{P_\rho^Y} S'_Y$$

Proof. In the matrix form $P_\rho^G = \frac{1}{|G|}1$ where 1 is the all ones matrix. Thus, for $v = \sum_g a_g e_g$, $(I - P_\rho^G)(v) = \sum_g \left(a_g - \frac{\sum_g a_g}{|G|}\right) e_g$. It's easy to see that the image is V'_{reg} and it surjects.

We have already shown that P_ρ^T is an orthogonal projection to the space of invariants of T for each $T = G, X, Y$. Thus, $(I - P_\rho^G)^2 = I - 2P_\rho^G + \left(P_\rho^G\right)^2 = I - P_\rho^G$.

P_ρ^X is clearly an orthogonal projection because we already know that P_ρ^X on V_{reg} is and here we remove a 1-d subspace from each such that the subspace removed from S_X is the image of P_ρ^X □

Corollary 13. *Composing the 2 maps above we get that the following map (say, ϕ) is an orthogonal projection.*

$$V_{reg} \oplus V_{reg} \xrightarrow{\phi = \begin{bmatrix} P_\rho^X(I - P_\rho^G) & 0 \\ 0 & P_\rho^Y(I - P_\rho^G) \end{bmatrix}} S'_X \oplus S'_Y \cong S'$$

Lemma 14. *The map $M_X = P_\rho^X\big|_{S'_X}$ Moreover, $\|M_X\|_* = \|P_\rho^X\big|_{S'_X}\|_2 = \|P_\rho^X\big|_{S'_Y}\|_2$*

Proof. From the explicit definition of M_i, we have $M_X : S'_X \to S'_Y$ such that $M_X(gX) = \frac{1}{d_X}\sum_{((gX,hY)\in E)} hY$ where d_X is the degree of gX. From the uniformity property, we know that, $(gX, hY) \in E \iff \exists g', (gX, hY) = (g'X, g'Y) \in E$. But, $X \cap Y = e \implies h = g' = gx\ x \in X$ Thus, the set of neighbours of gX can be written as $\{gxY \mid x \in X\}$. Thus,

$$\begin{aligned} M_X(gX) &= \frac{1}{|X|}\sum_{x\in X}(gx)Y \\ &= \left(\frac{1}{|X|}\sum_{x\in X}\rho(x)g\right) \\ &= P_{V_\rho^X}(g)Y \end{aligned}$$

Now the spectral norm of the projection matrices are defined over the usual Euclidean norm but we had defined a scaled norm for M_i. Denote the usual norm by $\|\|_2$ and the scaled norm by $\|\|_*$. We had that for V_X, $\|g\|_* = \sqrt{|G/X|}\|g\|_2$ Thus,

$$\begin{aligned} \|M_X\|_* &= \max_{v\in S'_X}\frac{\|M_x(v)\|_*}{\|v\|_*} \\ &= \max_{v\in S'_X}\frac{\sqrt{|G/Y|}\|M_x(v)\|_2}{\sqrt{|G/X|}\|v\|_2} \\ &= \max_{v\in S'_X}\frac{\|P_{V_\rho^X}(v)\|_2}{\|v\|_2} \quad \text{because in our case } |X| = |Y| \end{aligned}$$

The second equality just follows from the fact that $S'_X \cong S'_Y$ □

Theorem 15. *X, Y are ϵ-orthogonal iff $\|M'\|_* \leqslant \epsilon$*

Proof. From Theorem 9, we have ϵ-orthogonality iff $\|P_\rho^X(P_{V_\rho^Y} - P_{V_\rho^G})\|_2 \leqslant \epsilon$. Since each is an orthogonal projection, we have,

$$\begin{aligned} \|P_\rho^X(P_{V_\rho^Y} - P_{V_\rho^G})\|_2 &= \max_{v\in V_{reg}}\frac{\|P_\rho^X(P_{V_\rho^Y} - P_{V_\rho^G}(v))\|_2}{\|v\|_2} \\ &= \max_{v\in V_{reg}}\frac{\|P_{V_\rho^X}(P_{V_\rho^Y})(I - P_{V_\rho^G})(v)\|_2}{\|v\|_2} \\ &= \max_{w\in S'}\frac{\|P_{V_\rho^X}(P_{V_\rho^Y})(w)\|_2}{\|w\|_2} \quad \text{Since the projection is orthogonal } \|v\| = \|w\| \\ &= \max_{z\in S'_Y}\frac{\|P_{V_\rho^X}(z)\|_2}{\|z\|_2} \quad \text{Since the projection is orthogonal } \|z\| = \|w\| \\ &= \|P_\rho^X|_{S'_Y}\|_2 = \|M_X\|_* \quad \text{By the previous lemma} \end{aligned}$$

Thus, X, Y are ϵ-orthogonal iff $\|M'\|_* \leqslant \epsilon$ □

9 1-D links are ϵ-orthogonal

This is a result from [EJZ09] which proves the orthogonality on the way to prove Property(T) for $EL_n(\mathcal{R})$. It proves a much general version of this among other things but the proof of the case we need is fairly straightforward.

Theorem 16. *Given $H = EL_{n+1}(\mathbb{F}_q[t]/\langle t^s\rangle)$ and X, Y defined as before, X, Y are $\frac{1}{\sqrt{q}}$-orthogonal i.e. for any irreducible unitary representation ρ, $\theta(V_\rho^X, V_\rho^Y) \leqslant \frac{1}{\sqrt{q}}$*

Proof. Consider an irreducible unitary representation (V, ρ). Since $[X, Z] = [Y, Z] = 1$, the subgroup Z commutes with the entire group and thus $\rho(z)$ commutes with every $\rho(g)$. Hence by Schur's lemma, [6] $\rho(z) = \lambda I$

Let's assume ρ is injective. This is not necessarily true and we will return to this later. If $V^Y = 0$, then by symmetry so is V^X and we trivially have the orthogonality. So assume that's not the case. Let $0 \neq u \in V^Y$. Let $L = \mathsf{span}_{\mathbb{C}}(\rho(x)u \mid x \in X)$. L is clearly X and Z invariant i.e. $\forall v \in L, g \in X, Z,\ \rho(g)v \in L$. Now, $\rho(y)(\rho(x)u) = \rho(xy[x, y])u = \rho([x, y])\rho(xy)u = \lambda\rho(x)u \in L$. Thus, L is G invariant and since V is irreducible, $L = V$. To compute V^X, we take the image of $V = L$ under the projection P_ρ^X which gives us $V^X = \mathsf{span}_{\mathbb{C}}(u_x)$ where $u_X = \sum_{x \in X} \rho(x) \cdot u$. By symmetry V^Y is also one-dimensional and we already have $u \in V^Y$ and thus $V^Y = \mathsf{span}_{\mathbb{C}}(u)$.

Lemma 17. $\langle u, \rho(x)u\rangle = 0\ \forall x \in X\backslash\{1\}$. *And thus,* $\|u_x\| = \sqrt{|X|}\|u\|$

Proof. Let $e \neq y \in Y$. Now we have,

$$\begin{aligned}
\langle u, \rho(x)u\rangle &= \langle \rho(y)u, \rho(y)\rho(x)u\rangle && \text{Since } \rho \text{ is unitary} \\
&= \langle u, \rho(y)\rho(x)u\rangle && \text{Since } u \in V^Y,\ \rho(y)u = u \\
&= \langle u, \lambda\rho(x)u\rangle && \text{As computed above, } \lambda = \rho([x, y])
\end{aligned}$$

By our assumption of injectivity, $\lambda = 1$ only for the identity, but for any non-trivial y, $[x, y] \neq e$. Thus, $\langle u, \rho(x)u\rangle = 0$ and $\|u_x\|^2 = \langle \sum_{x \in X} \rho(x)u, \sum_{x \in X} \rho(x)u\rangle = |X|\langle u, u\rangle$ □

Since the representation is irreducible, $V^X \cap V^Y = V^G = (0)$. Thus, in the definition of angle, $V^{X'} = V^X$ Thus,

$$\begin{aligned}
\theta(V^X, V^Y) &= \min_{a \in V^Y, b \in V^X} \frac{\langle a, b\rangle}{\|a\|\|b\|} = \frac{\langle u, u_x\rangle}{\|u\|\|u_x\|} && \text{Since } V^X, V^Y \text{ are one-dimensional} \\
&= \sum_{x \in X} \frac{\langle u, \rho(x)u\rangle}{\|u\|\|u_x\|} = \frac{\langle u, u\rangle}{\sqrt{|X|}\|u\|\|u\|} = \frac{1}{\sqrt{|X|}} && \text{From the above lemma}
\end{aligned}$$

However, we had made the assumption that ρ is injective but if it is not we look at the image $\rho(G)$, now, we can run the argument again for this group and the representation being the mere inclusion into GL_n, thus we have $\frac{1}{\sqrt{|\rho(X)|}}$-orthogonality and we need to bound the smallest subgroup. If we define the Heisenberg group with $\mathcal{R} = \mathbb{F}_q[t]/t^s$ then a subgroup of X would be a subgroup of $\mathcal{R}$ and the smallest one is of size q which is just X with entries from $\mathbb{F}_q$. Therefore, $|\rho(X)| \geqslant q$ and we have $\frac{1}{\sqrt{q}}$-orthogonality. By the equivalence with spectral gap, it's second highest eigenvalue is also bounded above by it. □

[6] Given a G-representation (V, ρ), map $\phi : V \to V$ is G-linear if, $\forall g \in G, \rho(g)\phi(v) = \phi(\rho(g)(v))$. Schur's lemma states that the only G-linear maps between V, V are of the form λI. If g commutes with every element, then, $\rho(g)$ clearly satisfies the G-linearity conditions and thus by the lemma is λI

10 Wrapping it up

Let's now prove the other 2 requirements and tie it all together to give us the final result.

10.1 Infinite Vertices

The $|X^{(S)}(0)| = \sum_i |EL_{n+1}(\mathcal{R})/K_i| \geqslant n|G|/|K_0|$ From the explicit description of K_i, (see Theorem 3) it's easy to see that $\leqslant |K_i| = |T|^{(n-1)^2} = q^{2(n-1)^2}$ and $EL_{n+1}(\mathcal{R}) = |\mathcal{R}|^{n(n-1)/2} = |\mathbb{F}_q[t]/\langle t^s\rangle = q^{sn(n-1)/2}$. Thus, $|X^{(S)}(0)| = nq^{sn(n-1)/2-2(n-1)^2}$ which clearly tends to infinity.

10.2 Bounded Degree

We saw above that $|K_i| = q^{2(n-1)^2}$ is not a function of s and is a constant, say Q. Now, take an arbitrary vertex say, gK_i. By the uniformity lemma, each σ containing is of the form $\sigma = (hK_0, \cdots hK_n)$ such that $hK_i = gK_i$. Thus, $|\{\sigma \in X^{(s)}(n) \mid v \in \sigma\}| = |h \mid hK_i = gK_i| = |K_i| \leqslant Q$

10.3 Expansion

We have shown already Theorem 9 that all 1-d links expand with $\lambda \leqslant \frac{1}{\sqrt{q}}$. We now quote the result from [Opp18].

Theorem 18 ([Opp18]). *Given a pure n-dimensional strongly gallery connected simplicial complex X such that $\mu_\tau \leqslant \lambda$, $\forall \tau \in X(n-2)$, then $\mu_\tau \leqslant \frac{\lambda}{1-k\lambda}$ $\forall \tau \in X(n-2-k)$. In particular, X is a one-sided $\frac{\lambda}{1-(n-1)\lambda}$-local spectral expander.*

Thus, plugging in $\lambda = \frac{1}{\sqrt{q}}$, we get that X is a one-sided $\frac{1}{\sqrt{q}-n+1}$ local spectral expander.

10.4 Final Result

Theorem 19. *Let $n \geqslant 2$ and let q be a prime power such that $q > (n-1)^2$. For $s \in \mathbb{N}$, let $\mathbb{F}_q[t]/t^s$ to be the $\mathbb{F}_q$ algebra with the generating set $\{1, t\}$. Let $X(s)$ be the simplicial complex of the subgroup geometry system of $EL_{n+1}(\mathbb{F}_q[t]/t^s)$ Then for every $s > n$, the following holds for $X(s)$*

1. *$X(s)$ is a pure n-dimensional, (n+1)-partite, strongly gallery connected clique complex with no free faces.*
2. *$X(s)$ is finite and the number of vertices of $X(s)$ tends to infinity as s tends to infinity.*
3. *There is a constant $Q = Q(q)$ such that for every s, each vertex of $X(s)$ is contained in exactly Q n-dimensional simplices.*
4. *$X(s)$ is a $\frac{1}{\sqrt{q}-n+1}$ -local spectral expander.*

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