Saptharishi Simplified Analysis of Kaufman Oppenheim HDX Construction

October 14, 2019

1 Introduction

Ramprasad Saptharishi found a simpler expansion analysis of the Kaufman Oppenheim (KO) HDX construction [1]. The original KO analysis was based on a folklore representation theoretic result employed in the study of Cayley graphs. On the other hand, Saptharishi's analysis is elementary using only eigenvalue interlacing.

2 Recalling KO Construction

2.1 General Coset Geometry

We recall the KO construction. Given a group G and a collection of its cosets $K_1, \ldots, K_d$, we form the following simplicial complex $\mathfrak{X}(G, \{K_1, \ldots, K_d\})$ as

- $\mathfrak{X}(1)$ is the cosets of $K_1, \ldots, K_d$ in G, and
- $\{a_1 K_1, \ldots, a_d K_d\} \in \mathfrak{X}(d)$ if $a_1 K_1 \cap \ldots \cap a_d K_d \neq \emptyset$.

This kind of simplicial complex is also known as a *coset geometry*.

Fact 2.1. *The 1-skeleton of $\mathfrak{X}(G, \{K_1, \ldots, K_d\})$ is connected iff $G = \langle K_1, \ldots, K_d \rangle$.*

Fact 2.2. *Let $K_S = \bigcap_{i \in S} K_i$ and $K_\emptyset \coloneqq G$. Then all links are isomorphic to $\mathfrak{X}(K_S, \{K_{S \cup \{i\}} | i \notin S\})$.*

Corollary 2.2.1. *The 1-skeleton of all the links are connected if $K_S = \langle K_{S \cup \{i\}} \mid i \notin S \rangle$.*

2.2 KO Particular Group

For the particular choice of group G in KO construction, we will need elementary matrices with entries in $\mathcal{R} = \mathbb{F}_p[t]/\langle t^s \rangle$. For $r \in \mathcal{R}$, recall that the the elementary matrix $e_{i,j}(r) \in \mathcal{R}^{d \times d}$ is defined as follows

$$[e_{i,j}(r)]_{k,m} = \begin{cases} 1 & \text{if } k = m \\ r & \text{if } k = i \text{ and } m = j \\ 0 & \text{otherwise} \end{cases}$$

We define the cosets used in KO. We have

$$K_i \coloneqq \langle e_{j,j+1}(at + b) \mid a, b \in \mathbb{F}_p, j \neq i \rangle,$$

and more generally

$$K_S := \langle e_{j,j+1}(at+b) \mid a, b \in \mathbb{F}_p, j \notin S \rangle.$$

Fact 2.3. $K_S = \bigcap_{i \in S} K_i$.

The top links (those of co-dimension 2) are either complete bipartite graphs or a bipartite graph on bipartitions H_1 and H_2 of coset representatives.

$$H_1 := \left\{ \begin{pmatrix} 1 & 0 & Q \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix} \mid \ell \in \mathbb{F}_p[t]^{\leq 1}, Q \in \mathbb{F}_p[t]^{\leq 2} \right\}$$

$$H_2 := \left\{ \begin{pmatrix} 1 & \ell & Q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \ell \in \mathbb{F}_p[t]^{\leq 1}, Q \in \mathbb{F}_p[t]^{\leq 2} \right\}$$

For convenience, we represent a coset as $H_i(\ell_i, Q_i)$ for $i \in [2]$. The adjacency relation is given below.

Fact 2.4. *$H_1(\ell_1, Q_1) \sim H_2(\ell_2, Q_2)$ iff $\ell_1 \cdot \ell_2 = Q_1 + Q_2$.*

3 Expansion Analysis

The key realization of Saptharishi was to observe that the graphs appearing in the links of co-dimension two in KO construction (apart from the complete bipartite ones) are closely related to the following graph (featured in the Zig-Zag paper). Let $V = \mathbb{F}_q \times \mathbb{F}_q$ where $q = p^3$. We connect $(a, b), (c, d) \in V$ iff $ac = b + d$.

Claim 3.1. *G has degree q.*

Claim 3.2. *The second normalized largest singular value of G is $\lambda(G) \leq 1/\sqrt{q}$.*

Proof. Let A be the adjacency operator of G (not normalized). We analyze A^2. Note that $(a, b) \sim (c, d)$ in A^2 if there exist $(a', b') \in V(G)$ such that $(a, b) \sim (a', b')$ and $(a', b') \sim (c, d)$, i.e., $aa' = b+b'$ and $a'c = b' + d$ which implies $a'(a - c) = b - d$. Then, if $a = c$, we need $b = d$ for $(a, b) \sim (c, d)$ in A^2. Analogously, if $a \neq c$, we can always choose $a' = (b - d)/(a - c)$ and $b' = (cb - ad)/(a - c)$. Sorting the rows and columns in lexicographic order, we get

$$A^2 = \begin{pmatrix} qI & \dots & J \\ J & \ddots & J \\ J & \dots & qI \end{pmatrix} = qI + (J - I) \otimes J,$$

from which the claim readily follows. □

Now, consider the induced subgraph G' of G in which we only keep $(a, b) \in V$ such that $a = (a_1, a_2, a_3) \in \mathbb{F}_p^3$ with $a_1 = 0$. Denote by B the adjacency operator of G'.

Claim 3.3. *The second largest eigenvalue of B/p^2 is $1/\sqrt{p}$.*

Proof. By eigenvalue interlacing and accounting for normalizing factors, we conclude that the second largest eigenvalue of B/p^2 is at most

$$\frac{q \cdot 1/\sqrt{q}}{p^2} = \frac{1}{\sqrt{p}},$$

as claimed. $\square$

References

[1] Tali Kaufman and Izhar Oppenheim. Construction of new local spectral high dimensional expanders. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2018, pages 773–786. ACM, 2018.