Local Analysis of Higher-Order Random Walks and Applications

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1 Tighter Spectral Analysis

These notes are based on Vedat Levi Alev exposition in the HDX-Codes cluster.

Define the operators $P_k^{\bigtriangledown} \coloneqq U_{k-1} D_k$ and $P_k^{\bigtriangleup} \coloneqq D_{k+1} U_k$ where $U_k \colon \mathbb{R}^{X(k)} \to \mathbb{R}^{X(k+1)}$ and $D_k \colon \mathbb{R}^{X(k)} \to \mathbb{R}^{X(k-1)}$ are the usual up and down operators of a weighted simplicial complex $X(\leq d)$. For convenience, we will be working with size rather than dimension.

Theorem 1.1 (Kaufman-Oppenheim [3])**.**

$$\lambda_2(P_k^{\bigtriangledown}) \leq 1 - \frac{1}{k+1} + \frac{k}{2} \cdot \gamma,$$

where $\gamma = \max_{S \in X(\leq k-2)} \lambda_2(A_S)$*.*

Remark 1.2. *The above result requires the links to be* $\gamma = O(1/k^2)$ *expanding to give a non-trivial bound on* $\lambda_2(P_k^{\bigtriangledown})$*.*

Alev et al. show that the $P_k^{\bigtriangledown}$ walk satisfy $\lambda_2(P_k^{\bigtriangledown}) < 1$ provided $\gamma < 1$. More precisely, they showed the following.

Theorem 1.3 (Alev et al. [1])**.** *For* $k \in [d]$*, we have*

$$\lambda_2(P_{k-1}^{\bigtriangleup}) = \lambda_2(P_k^{\bigtriangledown}) \leq 1 - \frac{1}{k} \prod_{i=0}^{k-2} (1 - \gamma_i),$$

where $\gamma_i = \max_{S \in X(i)} \lambda_2(A_S)$ *and* A_S *is the normalized adjacency operator of the link* X_S*.*

Define and $P_k^{\wedge} \coloneqq (k+1)/k \cdot (D_k U_{k+1} - I/(k+1))$, i.e., $P_k^{\wedge}$ is the non-lazy version of $P_k^{\bigtriangleup}$. The key lemma is the following.

Lemma 1.4 (Key Lemma)**.** *For* $k \in [d-1]$*, we have*

$$P_k^{\wedge} - P_k^{\bigtriangledown} \preceq \gamma_{k-1}(I - P_k^{\bigtriangledown}).$$

A version of Lemma 1.4 was known with RHS $\gamma_{k-1} I$. The change to $\gamma_{k-1}(I - P_k^{\bigtriangledown})$ was crucial to the improved bound on $\lambda_2(P_k^{\bigtriangledown})$. We omit the proof of Lemma 1.4, but observe that it follows from a finer account of the mass of "parallel vectors on the links" similar in spirit to the proof of Oppenheim Trickling-down Theorem.

Proof of Theorem 1.3. The equality $\lambda_2(P_{k-1}^{\triangle}) = \lambda_2(P_k^{\triangledown})$ readily follows from $P_{k-1}^{\triangle} \coloneqq D_k U_{k-1}$ and $P_k^{\triangledown} \coloneqq U_{k-1} D_k$ having the same multiset of non-zero eigenvalues.

Now, the proof proceeds by induction. Note that $P_1^{\triangledown} = U_0 D_1$ is rank one and thus $\lambda_2(P_1^{\triangledown}) = 0$ proving the base case.

From Lemma 1.4, we have $P_k^{\wedge} \preceq \gamma_{k-1} I + (1-\gamma_{k-1}) P_k^{\triangledown}$ implying

$$\begin{aligned} \lambda_2(P_k^{\wedge}) &\leq \gamma_{k-1} + (1-\gamma_{k-1})\lambda(P_k^{\triangledown}) \\ &\leq \gamma_{k-1} + (1-\gamma_{k-1})(1 - \frac{1}{k}\prod_{i=0}^{k-2}(1-\gamma_i)) \\ &= 1 - \frac{1}{k}\prod_{i=0}^{k-1}(1-\gamma_i). \end{aligned}$$

Since $P_k^{\wedge} \coloneqq (k+1)/k \cdot (P_k^{\triangle} - I/(k+1))$, we concluded that

$$\lambda_2(P_k^{\triangle}) \leq 1 - \frac{1}{k+1}\prod_{i=0}^{k-1}(1-\gamma_i),$$

also giving a bound for $\lambda_2(P_{k+1}^{\triangledown}) = \lambda_2(P_k^{\triangle})$. □

2 Applications

For instance, fast mixing of the high-dimensional walks has found applications in counting/sampling combinatorial objects [2]. Therefore, it is highly desirable to reduce the expansion requirement of the links as much as possible.

References

[1] Vedat Levi Alev, Nima Anari, Lap Chi Lau, Kuikui Liu, and Shayan Oveis Gharan. A note on the second eigenvalue of higher order random walks. 2019. `https://cs.uwaterloo.ca/~vlalev/notes/higher-order.pdf`.

[2] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid. *CoRR*, abs/1811.01816, 2018.

[3] Tali Kaufman and Izhar Oppenheim. High order random walks: Beyond spectral gap. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2018, August 20-22, 2018 - Princeton, NJ, USA*, pages 47:1–47:17, 2018.